

Topologies of Stochastic Markov Models: Computational Aspects^{*}

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Abstract. In this paper we propose two behavioral distances that support approximate reasoning on Stochastic Markov Models (SMMs), that are continuous-time stochastic transition systems where the residence time on each state is described by a generic probability measure on the positive real line. In particular, we study the problem of measuring the behavioral dissimilarity of two SMMs against linear real-time specifications expressed as Metric Temporal Logic (MTL) formulas or Deterministic Timed-Automata (DTA).

The most natural choice for such a distance is the one that measures the maximal difference that can be observed comparing two SMMs with respect to their probability of satisfying an arbitrary specification. We show that computing this metric is **NP**-hard. In addition, we show that any algorithm that approximates the distance within a certain absolute error, depending on the size of the SMMs, is **NP**-hard.

Nevertheless, we introduce an alternative distance, based on the Kantorovich metric, that is an over-approximation of the former and we show that, under mild assumptions on the residence time distributions, it can be computed in polynomial time.

1 Introduction

Continuous-time probabilistic systems constitute the basic semantical tool to model random phenomena in complex real-time applications. They are successfully exploited in performance and dependability analysis and mostly used in applications such as systems biology, modeling/testing of cyber-physical systems, machine learning, and analysis of lossy network systems, etc.

In this context, the models are verified (or *model checked*) against real-time specifications aiming at determining the probability by which these are attained. In this way one may prove that unwanted behaviors are unlikely to occur (e.g., with a low probability) within a given time horizon or that certain events happen according to a specific desired timing pattern with high probability. Usually, real-time specifications are expressed as temporal logical formulas [21,2,3,5] or as the language recognized by automata-like formalisms such as Timed Automata

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(TAs) [1]. In this work we focus on the class of *linear* real-time specifications expressed either as Metric Temporal Logic (MTL) formulas [21,2,3] or as the timed-languages recognized by Deterministic Timed Automata (DTAs) [4]. Our attention on linear-time properties, opposed to branching-time properties, is motivated by the fact that in many applications the system to be modeled cannot be internally accessed, but only tested via observations performed over a set of random executions. For instance, this is mostly common in application domains such as systems biology, modeling/testing of cyber-physical systems, and machine learning.

However, when one aims at verifying properties of a real system out of its model representation, he should also take into consideration the degree of inaccuracy of the representation. Indeed, if the real-valued parameters of the model have been acquired from empirical data subject to error estimates, any analysis performed on it is itself subject to an inherent source of inaccuracy, that may lead to deceptive results on the original system. This motivated the study of *behavioral metric semantics*, initially developed for discrete-time Markov chains (MCs) [15], then extended to continuous-time models, generalized semi-Markov processes [19], and general Markov Processes [15], which provide for a formal notion of behavioral similarity between systems.

In this paper we study metrics aimed at helping the verification of continuous-time Markov systems against linear real-time specifications. In particular, we define two pseudometrics, respectively called *MTL* and *DTA variation distances*, that measure the maximal difference that can be observed by comparing two models with respect to their likelihood of satisfying any specification expressed in the form of an MTL formula or as a timed language recognized by a DTA, respectively. Therefore, knowing that two models are at distance $\epsilon \geq 0$ from each other ensures that any result obtained by an analysis w.r.t. a specification (hence, an MTL formula or a DTA) on one system can be reflected to the other with an absolute error bounded by ϵ .

The technical contributions of this paper can be summarized as follows.

1. We introduce *stochastic Markov models* (SMMs), that are continuous-time stochastic transition systems where the residence time on a state is specified by a generic probability measure on the positive real line. These generalize both MCs —where probabilistic transitions happens instantaneously— and time-homogeneous Continuous-Time Markov Chains (CTMCs) —where the residence time probability on states is characterized by a negative exponential distribution.
2. We study the measure-theoretical and topological properties of real-time specifications expressed as MTL formulas or DTAs. Specifically, we show that the σ -algebras generated, respectively, by MTL and DTA specifications, coincide. Then, we introduce a pseudometric between real-time specifications (that are, measurable sets of timed paths) and we single out a considerably simple family of specifications, namely those represented as resetting single-clock DTAs (1-RDTA), that is *dense* in the whole σ -algebra. This will imply that the probability measured in any measurable set can be approximated arbitrarily close by an 1-RDTAs. This has practical applications in quantitative model checking

on CTMCs, since this allows one to exploit efficient algorithms for *single clock* DTAs [12] to approximate model checking against DTAs or MTL formulas.

3. We consider two total variation distances on SMMs: one characterizing the maximal variation w.r.t. MTL specifications, the other w.r.t. DTA specifications. We show that these two pseudometrics coincide, and that the total variation can be obtained solely looking at 1-RDTAs specifications.

4. We prove that the problem of computing the distance, *exactly*, is **NP**-hard. This is done via a reduction from Max Clique, by generalizing an argument by Lyngsø and Pedersen [24]. Furthermore, we show that even the problem of approximating the distance within a certain absolute error, depending on the size of the SMMs, is **NP**-hard. To the best of our knowledge, whether the distance is computable or not still remains an open problem.

5. Nevertheless, we provide a fixed point bisimilarity distance, based on the Kantorovich metric, that is an over-approximation of the former. Then, extending a result by Chen et al. [10] for MCs, we prove that, under mild assumptions on the residence time distributions occurring in the SMMs, this distance can be computed in polynomial time using the ellipsoid algorithm.

The paper has an appendix that contains some of the proofs that could not be included in the paper due to their size or complexity.

2 Preliminaries

In this section we recall the basic notions used in the paper and fix the notation.

Measure theory. A *field* over a set X is a nonempty family \mathcal{F} of subsets of X closed under complement and finite union. A σ -*algebra* Σ over X is a field also closed under countable union. The pair (X, Σ) is called a *measurable space* and the elements of Σ *measurable sets*. For a family \mathcal{F} of subsets of X , the σ -algebra generated by \mathcal{F} is the smallest σ -algebra containing \mathcal{F} , denote by $\sigma(\mathcal{F})$.

Consider two measurable spaces (X, Σ_X) and (Y, Σ_Y) . A function $f: X \rightarrow Y$ is *measurable* if for all $E \in \Sigma_Y$, $f^{-1}(E) = \{x \mid f(x) \in E\} \in \Sigma_X$. The *product spaces*, $(X, \Sigma_X) \otimes (Y, \Sigma_Y)$, is the measurable space $(X \times Y, \Sigma_X \otimes \Sigma_Y)$, where $\Sigma_X \otimes \Sigma_Y$ is the σ -algebra generated by the *rectangles* $E \times F \in \Sigma_X \times \Sigma_Y$.

A *measure* on a measurable space (X, Σ) is a σ -additive function $\mu: \Sigma \rightarrow [0, \infty]$, i.e. $\mu(\bigcup_{E \in \mathcal{F}} E) = \sum_{E \in \mathcal{F}} \mu(E)$ for all countable families \mathcal{F} of pairwise disjoint measurable sets; it is a *probability measure* if, in addition, $\mu(X) = 1$. We denote by $\Delta(X, \Sigma)$ the set of probability measures on (X, Σ) , and by $\mathcal{D}(X) = \Delta(X, 2^X)$ the set of (*discrete*) *probability distributions*.

Given two measurable spaces (X, Σ_X) and (Y, Σ_Y) , and a measurable function $f: X \rightarrow Y$, any measure μ on (X, Σ_X) defines a measure $\mu \circ f^{-1}$ on (Y, Σ_Y) ; this operation is called *push forward*, denoted by $\mu \# f$.

Given two measures μ and ν , on (X, Σ_X) and (Y, Σ_Y) , respectively, we define the *product measure* $\mu \times \nu$ on $(X, \Sigma_X) \otimes (Y, \Sigma_Y)$, as the *unique* measure such that $(\mu \times \nu)(E \times F) = \mu(E) \cdot \nu(F)$, for arbitrary $E \in \Sigma_X$ and $F \in \Sigma_Y$. A measure ω on $(X, \Sigma_X) \otimes (Y, \Sigma_Y)$ is a *coupling* for (μ, ν) if $\omega(E \times Y) = \mu(E)$ and

$\omega(X \times F) = \nu(F)$, for arbitrary $E \in \Sigma_X$ and $F \in \Sigma_Y$; μ and ν are the *left* and the *right marginals* of ω . We denote by $\Omega(\mu, \nu)$ the set of couplings for (μ, ν) .

Throughout the paper $(\mathbb{R}_+, \Sigma_{\mathbb{R}_+})$, or simply \mathbb{R}_+ , will denote the measurable space of positive real numbers with zero with Borel σ -algebra.

Metric spaces. Given a set X , a function $d: X \times X \rightarrow \mathbb{R}_+$ is a *pseudometric* on X if $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, y) + d(y, z) \geq d(x, z)$, for arbitrary $x, y, z \in X$; it is a *metric* if, in addition, $d(x, y) = 0$ iff $x = y$. A pair (X, d) where d is a (pseudo)metric on X is called a *(pseudo)metric space*.

Given a measurable space (X, Σ) , we consider two metrics on $\Delta(X, \Sigma)$:

- the *total variation distance*, defined for arbitrary $\mu, \nu \in \Delta(X, \Sigma)$ by

$$\|\mu - \nu\|_{\text{TV}} = \sup_{E \in \Sigma} |\mu(E) - \nu(E)|,$$

- the *Kantorovich (pseudo)metric*, defined for a (pseudo)metric d on X by

$$\mathcal{K}_d(\mu, \nu) = \inf \left\{ \int d \, d\omega \mid \omega \in \Omega(\mu, \nu) \right\}.$$

The space of timed paths. A *timed path* over a set X is an alternating infinite sequence $\pi = x_0, t_0, x_1, t_1, \dots$ of elements $x_i \in X$ and *time delays* $t_i \in \mathbb{R}_+$, for $i \in \mathbb{N}$. $\Pi(X)$ denotes the set of timed paths over X . For arbitrary $i \in \mathbb{N}$, let $\pi[i] = x_i$, $\pi\langle i \rangle = t_i$, $\pi|_i = x_0, t_0, \dots, t_{i-1}, x_i$ and $\pi|_i = x_i, t_i, x_{i+1}, t_{i+1}, \dots$. For $X_i \subseteq X$, $R_i \subseteq \mathbb{R}_+$, $i = 0..n$, let $\mathfrak{C}(X_0, R_0, \dots, R_{n-1}, X_n)$ be the *cylinder set* of the timed paths $\pi \in \Pi(X)$ such that $\pi|_n \in X_0 \times R_0 \times \dots \times R_{n-1} \times X_n$.

For (X, Σ) a measurable space, $\Pi(X, \Sigma)$ is the measurable space of timed paths over X , with σ -algebra $\Sigma_{\Pi(X, \Sigma)}$ generated by the *measurable cylinders* $\mathfrak{C}(S_0, R_0, \dots, R_{n-1}, S_n)$, where $S_i \in \Sigma$, $R_i \in \Sigma_{\mathbb{R}_+}$, $i = 0..n$, and $n \in \mathbb{N}$. If $\Sigma = \sigma(\mathcal{F})$ and $\Sigma_{\mathbb{R}_+} = \sigma(\mathcal{I})$, then $\Sigma_{\Pi(X, \Sigma)} = \sigma(\mathfrak{C}(\mathcal{F}, \mathcal{I}))$, where $\mathfrak{C}(\mathcal{F}, \mathcal{I})$ is the family of cylinders $\mathfrak{C}(F_0, I_0, \dots, F_{n-1}, I_n)$ where $F_i \in \mathcal{F}$, $I_i \in \mathcal{I}$, $i = 0..n$, and $n \in \mathbb{N}$. Moreover, if both \mathcal{F} and \mathcal{I} are fields, so is $\mathfrak{C}(\mathcal{F}, \mathcal{I})$.

For a function $f: X \rightarrow Y$, we define $f^\omega: \Pi(X) \rightarrow \Pi(Y)$ as the obvious stepwise extension of f on timed paths. Note that if f is measurable, so is f^ω .

3 Stochastic Markov Models

In this section we introduce the class of *Stochastic Markov Models* (SMMs) and define behavioral equivalences among them. Let AP be a countable set of *atomic propositions*, that we fix for the rest of the paper.

Definition 1 (Stochastic Markov Model). A stochastic Markov model is a tuple $\mathcal{M} = (S, A, \tau, \rho, \ell)$ consisting of a finite nonempty set S of states, a set $A \subseteq S$ of absorbing states, a transition probability function $\tau: S \setminus A \rightarrow \mathcal{D}(S)$, an exit-time probability function $\rho: S \setminus A \rightarrow \Delta(\mathbb{R}_+)$, and a labelling function $\ell: S \rightarrow 2^{AP}$.

The operational behavior of $\mathcal{M} = (S, A, \tau, \rho, \ell)$ can be described as follows: if the system is in state $s \in S$ and s is absorbing, no transition can be made; otherwise, it moves to an arbitrary $s' \in S$ within time $t \in \mathbb{R}_+$ with probability $\rho(s)([0, t]) \cdot \tau(s)(s')$. An atomic proposition $p \in AP$ is said to hold in s iff $p \in \ell(s)$.

SMMs subsume both MCs and *time-homogeneous* CTMCs. Indeed, MCs are the SMMs such that $A = \emptyset$ and, for all $s \in S$, $\rho(s)$ is the Dirac measure at 0 (transitions happen instantaneously); CTMCs are the SMMs such that, for all $s \notin A$, $\rho(s) = \text{Exp}(\lambda)$, where $\text{Exp}(\lambda)$ denotes the negative exponential distribution with parameter $\lambda > 0$.

An SMM $\mathcal{M} = (S, A, \tau, \rho, \ell)$ induces an S -indexed family of probability measures on the measurable space $\Pi(S, 2^S)$ of timed paths over S as follows.

Definition 2. Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM and $s \in S$. The probability measure $\mathbb{P}_s^{\mathcal{M}}$ on the measurable space of timed paths over S is the unique measure such that, for all $n \in \mathbb{N}$, $s_i \in S$ and $R_i \in \Sigma_{\mathbb{R}_+}$, $i = 0..n+1$,

$$\mathbb{P}_s^{\mathcal{M}}(\mathfrak{C}(s_0)) = \chi_{\{s\}}(s_0),$$

$$\mathbb{P}_s^{\mathcal{M}}(\mathfrak{C}(s_0, R_0, \dots, R_n, s_{n+1})) = \mathbb{P}_s^{\mathcal{M}}(\mathfrak{C}(s_0, R_0, \dots, R_{n-1}, s_n)) \cdot P(s_n, R_n, s_{n+1}),$$

where χ_E is the characteristic function of E and, for $s, s' \in S$, $R \in \Sigma_{\mathbb{R}_+}$, $P(s, R, s') = \rho(s)(R) \cdot \tau(s)(s')$, if $s \notin A$, and $P(s, R, s') = 0$ otherwise.

The existence of this measure is guaranteed by the Hahn-Kolmogorov extension theorem; uniqueness is guaranteed since, for all $s \notin A$, $\tau(s)$ and $\rho(s)$ are σ -finite. Intuitively, $\mathbb{P}_s^{\mathcal{M}}$ describes the probability that a stochastic run of \mathcal{M} starting from s belongs to a measurable set of $\Pi(S, 2^S)$.

Next we introduce two important behavioral equivalences on SMMs: *stochastic trace equivalence* and *bisimilarity*. To do so, for an SMM $\mathcal{M} = (S, A, \tau, \rho, \ell)$, we first define the following equivalence relations on S :

- $s \equiv_\ell s'$ if and only if $\ell(s) = \ell(s')$;
- $s \equiv_A s'$ if and only if, either $s, s' \in A$ or $s, s' \notin A$.
- $s \equiv s'$ if and only if $s \equiv_\ell s'$ and $s \equiv_A s'$.

For an SMM $\mathcal{M} = (S, A, \tau, \rho, \ell)$ we define $\mathcal{T}_{\mathcal{M}} = \mathfrak{C}(S/\equiv_\ell, \Sigma_{\mathbb{R}_+})$ and its elements will be called *trace cylinders*.

Definition 3 (Stochastic Trace Equivalence). Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM. Two states $s, s' \in S$ are stochastic trace equivalent with respect to \mathcal{M} , written $s \approx_{\mathcal{M}} s'$, if for all trace cylinders $T \in \mathcal{T}_{\mathcal{M}}$, $\mathbb{P}_s^{\mathcal{M}}(T) = \mathbb{P}_{s'}^{\mathcal{M}}(T)$.

Clearly $\approx_{\mathcal{M}} \subseteq \equiv_\ell$, moreover since states are also tested with respect to their associated probability on timed paths, we also have $\approx_{\mathcal{M}} \subseteq \equiv_A$.

Definition 4 (Bisimulation). Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM. An equivalence relation $R \subseteq S \times S$ is a bisimulation on \mathcal{M} if whenever $(s, s') \in R$,

- $s \equiv s'$; and
- if $s, s' \notin A$, then $\rho(s) = \rho(s')$ and, for all $C \in S/R$, $\tau(s)(C) = \tau(s')(C)$.

Two states $s, s' \in S$ are bisimilar with respect to \mathcal{M} , written $s \sim_{\mathcal{M}} s'$, if they are related by some bisimulation on \mathcal{M} .

These conditions require that any two bisimilar states are equally labelled and have identical probability w.r.t. any time delay of moving to any bisimilarity equivalence class. Note that, the above definition extends the notion of probabilistic bisimulation both on MCs [22] and CTMCs in [9].

4 Pseudometrics for Linear Real-Time Specifications

We start by introducing the two kinds of linear real-time specifications that we consider throughout the paper.

Metric Temporal Logic. Metric Temporal Logic (MTL) [2,3] has been introduced as a formalism for reasoning on sequences of events in a real-time setting. The grammar of logical formulas of MTL is as follows

$$\varphi ::= p \mid \perp \mid \varphi \rightarrow \varphi \mid \mathbf{X}^{[t,t']}\varphi \mid \varphi \mathbf{U}^{[t,t']}\varphi,$$

where $p \in AP$ and $t, t' \in \mathbb{Q}_+$ with $t \leq t'$.

Following [25], the semantics of MTL formulas φ is given by means of a satisfiability relation $\mathcal{M}, \pi \models \varphi$, defined, for an SMM $\mathcal{M} = (S, A, \tau, \rho, \ell)$ and a timed path $\pi \in \Pi(S)$, as follows.

$$\begin{aligned} \mathcal{M}, \pi \models p & \quad \text{if } p \in \ell(\pi[0]), \\ \mathcal{M}, \pi \models \perp & \quad \text{never,} \\ \mathcal{M}, \pi \models \varphi \rightarrow \psi & \quad \text{if } \mathcal{M}, \pi \models \psi \text{ whenever } \mathcal{M}, \pi \models \varphi, \\ \mathcal{M}, \pi \models \mathbf{X}^{[t,t']}\varphi & \quad \text{if } \pi(0) \in [t, t'], \text{ and } \mathcal{M}, \pi|_1 \models \varphi, \\ \mathcal{M}, \pi \models \varphi \mathbf{U}^{[t,t']}\psi & \quad \text{if } \exists i > 0 \text{ such that } \sum_{k=0}^{i-1} \pi\langle i \rangle \in [t, t'], \mathcal{M}, \pi|_i \models \psi, \\ & \quad \text{and } \mathcal{M}, \pi|_j \models \varphi \text{ whenever } 0 \leq j < i. \end{aligned}$$

The above is usually referred to as the *point-based* semantics. A key observation about this interpretation of formulas is that temporal connectives quantify over a countable set of positions in a timed path. In contrast, the *interval-based* semantics, adopted e.g., in [11,27], associates a state to each point in real time, and the temporal connectives quantify over the whole timed domain.

For $\varphi \in \text{MTL}$ we denote by $\llbracket \varphi \rrbracket_{\mathcal{M}} = \{\pi \in \Pi(S) \mid \mathcal{M}, \pi \models \varphi\}$ the set of all timed paths satisfying φ in \mathcal{M} , and define $\llbracket \text{MTL} \rrbracket_{\mathcal{M}} = \{\llbracket \varphi \rrbracket_{\mathcal{M}} \mid \varphi \in \text{MTL}\}$.

The next lemma states that the sets of timed paths satisfying a given MTL formula are measurable in $\Pi(S, 2^S)$. This justifies that MTL formulas can be used as linear real-time specifications for an SMMs.

Lemma 5. *Let \mathcal{M} be an SMM, then $\llbracket \text{MTL} \rrbracket_{\mathcal{M}} \subseteq \Sigma_{\Pi(S, 2^S)}$.*

Deterministic Timed Automata. Timed Automata (TAs) [1] have been introduced to model the behavior of real-time systems over time. Here we consider their deterministic variant without location invariants.

Let \mathcal{X} be a finite set of \mathbb{R}_+ -valued variables, called *clocks*, and let $\mathcal{V}(\mathcal{X})$ be the set of all valuations $v: \mathcal{X} \rightarrow \mathbb{R}_+$ for the clocks in \mathcal{X} . For $v \in \mathcal{V}(\mathcal{X})$, $t \in \mathbb{R}_+$, and $X \subseteq \mathcal{X}$, we denote by $\mathbf{0}$, the constant zero valuation, by $v + t$, the t -delay of v , and by $v[X := t]$, the update of X in v , all defined in the obvious way.

A *clock guard* $g \in \mathcal{G}(\mathcal{X})$ over \mathcal{X} is a finite set of expressions of the form $x \bowtie q$, for $x \in \mathcal{X}$, $q \in \mathbb{Q}_+$, and $\bowtie \in \{<, \leq, >, \geq\}$. We say that a valuation $v \in \mathcal{V}(\mathcal{X})$ *satisfies* a clock guard $g \in \mathcal{G}(\mathcal{X})$, written $v \models g$, if $v(x) \bowtie n$ holds, for all $x \bowtie q \in g$; two clock guards $g, g' \in \mathcal{G}(\mathcal{X})$ are *orthogonal* (or *non-overlapping*), written $g \perp g'$, if there is no $v \in \mathcal{V}(\mathcal{X})$ such that $v \models g$ and $v \models g'$.

Definition 6 (Deterministic Timed Automata). A deterministic timed automaton over a set of clocks \mathcal{X} is a tuple $\mathcal{A} = (Q, L, q_0, F, \rightarrow)$ consisting of a finite set Q of locations, a set L of symbols, an initial location $q_0 \in Q$, a set $F \subseteq Q$ of final locations, and a transition relation $\rightarrow \subseteq Q \times L \times \mathcal{G}(\mathcal{X}) \times 2^{\mathcal{X}} \times Q$ such that, whenever $(q, a, g, X, q'), (q, a, g', X', q'') \in \rightarrow$ and $g \neq g'$, then $g \perp g'$.

An run of a DTA $\mathcal{A} = (Q, L, q_0, F, \rightarrow)$ over a timed path $\pi = a_0, t_0, a_1, t_1, \dots$ over L , is an infinite sequence of the form

$$r = (q_0, v_0) \xrightarrow{a_0, t_0} (q_1, v_1) \xrightarrow{a_1, t_1} (q_2, v_2) \xrightarrow{a_2, t_2} \dots$$

with $q_i \in Q$ and $v_i \in \mathcal{V}(\mathcal{X})$, for all $i \geq 0$, satisfying the following requirements: (initialization) $v_0 = \mathbf{0}$; (consecution) for all $i \geq 0$, $v_{i+1} = (v_i + t_i)[X_i := 0]$, for some $(q_i, a_i, g_i, X_i, q_{i+1}) \in \rightarrow$ such that $v_i + t_i \models g_i$.

A run as above is *accepting* if $q_i \in F$, for some $i \geq 0$, and we say that π is *accepted* by \mathcal{A} . $\mathcal{L}(\mathcal{A})$ denotes the collection of all timed paths accepted by \mathcal{A} .

Observe that, due to the condition imposed on the transition relation, a deterministic timed automaton has at most one accepting run over a given timed path in $\Pi(L)$. Moreover, differently from TAs, which are only closed under finite union and intersection, DTAs are also closed under complement [1].

Following [12], a DTA accepting input symbols in 2^{AP} can be thought of as a linear real-time specification for SMMs. Formally, let $\text{DTA}(L)$ denote the collection of DTAs accepting symbols in L , then, for $\mathcal{A} \in \text{DTA}(2^{AP})$ and an SMM $\mathcal{M} = (S, A, \tau, \rho, \ell)$, we define $\llbracket \mathcal{A} \rrbracket_{\mathcal{M}} = \{\pi \in \Pi(S) \mid \ell^\omega(\pi) \in \mathcal{L}(\mathcal{A})\}$ as the set of all timed paths in \mathcal{M} accepted by \mathcal{A} , and $\llbracket \text{DTA} \rrbracket_{\mathcal{M}} = \{\llbracket \mathcal{A} \rrbracket_{\mathcal{M}} \mid \mathcal{A} \in \text{DTA}(2^{AP})\}$.

The next lemma justifies the use of DTAs as specifications for SMMs.

Lemma 7. Let \mathcal{M} be an SMM, then $\llbracket \text{DTA} \rrbracket_{\mathcal{M}} \subseteq \Sigma_{\Pi(S, 2^S)}$.

In the rest of the paper DTAs will be used only as specification on SMMs, so that $\text{DTA}(2^{AP})$ will be simply denoted by DTA.

4.1 Topological properties of MTL and DTA specifications

In this section we analyze the measure-theoretical and topological properties of MTL and DTA specifications.

The next lemma states that MTL and DTA specifications generate the same σ -algebra. Intuitively, two type of linear real-time specifications we consider, after being completed under complement and countable union, have the same expressivity. Moreover, the same σ -algebra can be generated by trace cylinders.

Lemma 8. Let \mathcal{M} be an SMM. Then $\sigma(\llbracket \text{MTL} \rrbracket_{\mathcal{M}}) = \sigma(\llbracket \text{DTA} \rrbracket_{\mathcal{M}}) = \sigma(\mathcal{T}_{\mathcal{M}})$.

From now on the σ -algebras in Lemma 8 will be simply referred as $\Sigma_{\mathcal{M}}$.

Now we consider the topological properties of $\Sigma_{\mathcal{M}}$. Let (X, Σ) be a measurable space, then any measure μ over it induces a pseudometric $d_\mu: \Sigma \times \Sigma \rightarrow \mathbb{R}_+$ on Σ , a.k.a. the *Fréchet-Nikodym pseudometric w.r.t. μ* , defined, for $E, F \in \Sigma$, by $d_\mu(E, F) = \mu(E \triangle F)$, where \triangle is the symmetric difference between sets¹.

¹ Triangular inequality follows by monotonicity and sub-additivity of μ noticing that, $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$

Lemma 9. *Let (X, Σ) be a measurable space and μ be a finite measure on it. If Σ is generated by a field \mathcal{F} , then \mathcal{F} is dense in the pseudometric space (Σ, d_μ) .*

Note that Lemma 9 is generic both in the field and in the measure that are given. In particular, since DTAs are closed under all Boolean operations [1], $\llbracket \text{DTA} \rrbracket_{\mathcal{M}}$ forms a field of sets; the same holds for $\llbracket \text{MTL} \rrbracket_{\mathcal{M}}$. Hence, we have the following.

Corollary 10. *Let $M = (S, A, \tau, \rho, \ell)$ be an SMM, and $s \in S$. Then, $\llbracket \text{MTL} \rrbracket_{\mathcal{M}}$ and $\llbracket \text{DTA} \rrbracket_{\mathcal{M}}$ are dense in $(\Sigma_{\mathcal{M}}, d_{\mathbb{P}_s^{\mathcal{M}}})$.*

Single-clock Resetting DTAs. The problem of model checking CTMCs against TA specifications is known to be computationally very hard, even restricting to the subclass of DTAs. Recently, Chen et al. [12] provided an algorithm that is efficient for *single-clock* DTAs. In this view, we show that the subclass of *resetting single-clock* DTAs (1-RDTAs) (i.e., DTAs with a single clock that is reset whenever a transition to the next location occurs) can be used to approximate with arbitrary precision any DTA or MTL specification. Indeed, 1-RDTAs are closed under Boolean operations², thus to use Lemma 9 it only remains to show the following.

Lemma 11. *$\llbracket 1\text{-RDTA} \rrbracket_{\mathcal{M}}$ is a generator for $\Sigma_{\mathcal{M}}$.*

Theorem 12 (1-RDTA-approximant). *Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM and $s \in S$. Then, for any $E \in \Sigma_{\mathcal{M}}$ and any $\epsilon > 0$, there exists $\mathcal{A} \in 1\text{-RDTA}$ such that $|\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_s^{\mathcal{M}}(\llbracket \mathcal{A} \rrbracket_{\mathcal{M}})| < \epsilon$.*

Theorem 12 allows one to use the algorithm of [12], to deploy approximate model checking of CTMCs against any real-time specification that is measurable in $\Sigma_{\mathcal{M}}$. We recall that this is the case both for DTAs and MTL formulas.

4.2 MTL and DTA Variation Pseudometrics

We consider distances on SMMs, specifically, variation pseudometrics parametric on the family of specifications where the maximal difference is meant to be tested.

Definition 13 (Variation distance). *Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM and $\mathcal{F} \subseteq \Sigma_{\Pi(S, 2^S)}$. We define the \mathcal{F} -variation pseudometric $\delta_{\mathcal{F}}^{\mathcal{M}}: S \times S \rightarrow [0, 1]$ as*

$$\delta_{\mathcal{F}}^{\mathcal{M}}(s, s') = \sup_{E \in \mathcal{F}} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)|.$$

Lemmas 5 and 7 justify to consider the variation pseudometric w.r.t. MTL and DTA specifications, i.e., $\delta_{\text{MTL}}^{\mathcal{M}}$ and $\delta_{\text{DTA}}^{\mathcal{M}}$, respectively³. In particular, as we have already done in the case of specifications (Lemma 8) we would like to compare the expressivity of the two metrics.

² Closure under union follows by the standard product construction, noticing that duplications of clocks are needed only to ensure the right resetting of the clocks.

³ Formally, the two distances should be denoted as $\delta_{\llbracket \text{MTL} \rrbracket_{\mathcal{M}}}^{\mathcal{M}}$ and $\delta_{\llbracket \text{DTA} \rrbracket_{\mathcal{M}}}^{\mathcal{M}}$, however the simplified notation will not cause any problem.

Lemma 14. *Let (X, Σ) be a measurable space and μ, ν be two finite measures on it. If Σ is generated by a field \mathcal{F} , then $\|\mu - \nu\|_{TV} = \sup_{E \in \mathcal{F}} |\mu(E) - \nu(E)|$.*

At this point we can state the main result of this section.

Theorem 15. *Let \mathcal{M} be an SMM. Then $\delta_{\text{MTL}}^{\mathcal{M}} = \delta_{\text{DTA}}^{\mathcal{M}} = \delta_{\Sigma_{\mathcal{M}}}^{\mathcal{M}}$.*

Now we will study the property of $\delta_{\Sigma_{\mathcal{M}}}^{\mathcal{M}}$, which will be simply referred as $\delta^{\mathcal{M}}$ (or δ) in the following. The first property, is that $\delta^{\mathcal{M}}$ is actually a behavioral distance in the sense that its kernel coincide with stochastic trace equivalence.

Theorem 16. *Let \mathcal{M} be an SMM. Then $\approx_{\mathcal{M}} = \{(s, s') \mid \delta^{\mathcal{M}}(s, s') = 0\}$.*

The next corollary is an immediate consequence of Lemmas 14 and 11 noticing that 1-RDTAs are closed under all Boolean operations.

Corollary 17. *Let \mathcal{M} be an SMM, then $\delta^{\mathcal{M}} = \delta_{1\text{-RDTA}}^{\mathcal{M}}$.*

This result is quite important, since it means that one can *exactly* determine the variational distance w.r.t. MTL and DTA specifications only looking at the subclass of 1-RDTA specifications, for which we already observed that quantitative model checking problem admits efficient computational solutions [12].

5 NP-Hardness and Inapproximability

In this section we show that computing δ is **NP**-hard. In addition, we prove that, for some ϵ depending on the size of the model, even the problem of approximating δ within an absolute error ϵ is **NP**-hard.

To this end we identify a subclass of SMMs where the total variation distance is characterized in terms of an L_1 distance over a suitable sub- σ -algebra of $\Sigma_{\mathcal{M}}$, generated by a family of cylinders in $\mathcal{T}_{\mathcal{M}}$ that we called *word cylinders sets*.

Definition 18 (Word cylinders). *Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM. We define $\mathcal{W}_{\mathcal{M}} = \mathfrak{C}(S/\equiv_{\ell}, \{\mathbb{R}_+\})$, and its elements words cylinder.*

Note that, the word cylinder sets are pairwise disjoint and, since the set of states is assumed to be finite, $\mathcal{W}_{\mathcal{M}}$ has countably many elements. This means that any $E \in \sigma(\mathcal{W}_{\mathcal{M}})$ can be expressed as a countable union of word cylinder sets.

Under the assumption that the residence time distributions that occur in \mathcal{M} are all equal, we can characterize $\delta^{\mathcal{M}}$ in terms of the L_1 distance between probability distributions in the measurable space $(\Pi(S), \sigma(\mathcal{W}_{\mathcal{M}}))^4$.

Lemma 19. *Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM such that $\rho(s) = \rho(s')$ for all $s, s' \notin A$, then $2 \cdot \delta^{\mathcal{M}}(s, s') = \sum_{E \in \mathcal{W}_{\mathcal{M}}} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)|$.*

⁴ We recall that for $\nu, \mu \in \mathcal{D}(X)$, $L_p(\mu, \nu) = (\sum_{x \in X} |\mu(x) - \nu(x)|^p)^{1/p}$.

The previous correspondence allows us to prove that computing δ on generic SMMs is **NP**-hard. The proof is carried out following an argument similar to Lyngsø and Pedersen [24], who proved the **NP**-hardness of comparing hidden Markov models (HMM) w.r.t. the L_1 norm. Namely, we show that the size of the maximum clique in an undirected graph with n vertices can be computed within the time it takes to solve a Toeplitz system⁵ with n unknowns and constant terms obtained from computing the distance δ for some SMMs that can be constructed in polynomial time in the size⁶ of the input graph.

Theorem 20 (NP-hardness). *Computing δ over SMMs is **NP**-hard.*

The proof of Theorem 20 makes use of SMMs that share the same residence time distribution on each state and it is generic in this choice⁷. This implies that the hardness result holds also within MCs or CTMCs.

Corollary 21. *Computing δ over MCs or CTMCs is **NP**-hard.*

Recently, Cortes et al. [13] proposed a reduction similar to that of [24] to prove that computing the L_{2p+1} , for any $p \in \mathbb{N}$, between probabilistic automata is **NP**-hard to approximate within some absolute error that depends on the size of the given automata. Here we slightly generalize this idea on SMMs.

To this end, we first introduce some notation. Let $P: X \rightarrow \mathbb{R}$, we say that the algorithm A approximates P , respectively

- within an absolute error $\epsilon > 0$, if for all $\forall x \in X. |A(w) - P(w)| \leq \epsilon$; or
- within a (multiplicative) factor $\alpha > 1$, if $\forall x \in X. P(w) \leq A(w) \leq \alpha \cdot P(w)$.

Proposition 22. *Let \mathcal{M} be an SMM, and $\alpha > 1$. If there exists a polynomial-time algorithm that approximates $\delta^{\mathcal{M}}$ within an absolute error $\epsilon = f(\alpha, \text{size}(\mathcal{M}))$, for some $f: \mathbb{R} \times \mathbb{N} \rightarrow [0, 1]$, then there exists a polynomial-time algorithm that approximates Max Clique within a factor α .*

Proposition 22 makes any inapproximability result for Max Clique to turn into an inapproximability result for the problem of computing δ . A famous result by Feige et al. [16] states that any algorithm that approximates Max Clique within any constant factor is **NP**-hard. As a corollary of that, we get the following.

Corollary 23 (Inapproximability). *Given an SMM \mathcal{M} , there exists $\epsilon \in (0, 1]$, depending on the size of \mathcal{M} , such that any algorithm that approximates $\delta^{\mathcal{M}}$ within an absolute error ϵ is **NP**-hard.*

⁵ A Toeplitz system is a linear system of equations where the coefficient matrix has each descending diagonal from left to right constant. Toeplitz systems can be efficiently solved in time $\Theta(n^2)$ using the Levinson-Durbin procedure.

⁶ We denote by $\text{size}(X)$ the representation of an object X . In particular, rational numbers are represented as quotient of integers written in binary.

⁷ The only restriction consists in its representation, that has to be such that the construction made in Theorem 20 is polynomial in the size of the starting graph.

6 A Polynomially Computable Upper-Bound

In the literature, the problem of estimating the total variation distance is approached from two different perspectives. One consists in computing sharp estimates on suitable classes of distributions [20]; the second leverages on a well known relation between the total variation and the Kantorovic metric. In this work we pursue the second way, leading to the definition of an over-approximation of δ that enjoys two good properties: (a) it is computable in polynomial time, and (b) it is a pseudometric. In particular, the latter property allows one to translate any convergence obtained with the over-approximation into a convergence with respect to the trace pseudometric.

This goal is achieved in three steps: (i) we construct an over-approximation based on the notion of coupling model; (ii) we give a fixed point characterization of this over-approximation showing that it is a Kantorovich-based bisimilarity distance; (iii) we show that, under mild assumptions on the residence time distributions, the fixed point is computable in polynomial time.

6.1 An Over-Approximation

Coupling characterization. The construction of the over-approximation of δ will be based on a classic duality between the total variation distance of probability measures and their couplings (see [23, Theorem 5.2]), which states that: *the maximal variation in the probabilities, evaluated among all the measurable sets, corresponds to the minimal discrepancy measured among all the possible couplings.* Formally, the discrepancy associated to a coupling $\omega \in \Omega(\mu, \nu)$ for two measures $\mu, \nu \in \Delta(X, \Sigma)$, is the value $\omega(\neq)$ associated to the measurable set $\neq \in \Sigma \otimes \Sigma$, where $\neq = \{(x, y) \mid x \neq y \in X\}$. Since δ is a relaxation of the total variation distance, also the notion of discrepancy is relaxed accordingly.

Definition 24. Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM. We define $\equiv_\ell^\omega \subseteq \Pi(S) \times \Pi(S)$ as $\pi \equiv_\ell^\omega \pi'$ if and only if $\ell^\omega(\pi) = \ell^\omega(\pi')$.

Intuitively, \equiv_ℓ^ω is the stepwise extension of \equiv_ℓ to timed paths in the following sense: $\pi \equiv_\ell^\omega \pi'$ if $\pi[i] \equiv_\ell \pi'[i]$ and $\pi\langle i \rangle = \ell\pi'\langle i \rangle$ for all $i \in \mathbb{N}$.

Lemma 25 and 26 justify \equiv_ℓ^ω as the adequate notion for separability in $\Sigma_{\mathcal{M}}$.

Lemma 25 (Separability). Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM and $\pi, \pi' \in \Pi(S)$. Then, $\pi \equiv_\ell^\omega \pi'$ iff [for all $C \in \Sigma_{\mathcal{M}}$, $\pi \in C$ iff $\pi' \in C$].

Lemma 26. Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM, then $\neq_\ell^\omega \in \Sigma_{\mathcal{M}} \otimes \Sigma_{\mathcal{M}}$.

The behavioral analogue of the duality between the total variation distance between probability measures and their couplings is generalized in $\Sigma_{\mathcal{M}}$ as follows.

Lemma 27. Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM and $s, s' \in S$, then

$$\sup_{E \in \Sigma_{\mathcal{M}}} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)| = \min \{ \omega(\neq_\ell^\omega) \mid \omega \in \Omega(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}}) \} .$$

The characterization given by Lemma 27 suggests us to over-approximate δ by restricting the couplings where to search for the minimal discrepancy.

Definition 28 (Coupling models). Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM. A coupling model for \mathcal{M} is a tuple $\mathcal{C} = (S', A', \tau', \rho', \ell')$ such that

- i. $S' = S \times S$ and $A' = (A \times S) \cup (S \times A)$;
- ii. $\tau' : S' \setminus A' \rightarrow \mathcal{D}(S')$ and for all $s, s' \notin A$, $\tau'(s, s') \in \Omega(\tau(s), \tau(s'))$;
- iii. $\rho' : S' \setminus A' \rightarrow \Delta(\mathbb{R}_+ \times \mathbb{R}_+)$ and for all $s, s' \notin A$, $\rho'(s, s') \in \Omega(\rho(s), \rho(s'))$;
- iv. $\ell' : S' \rightarrow 2^{AP} \times 2^{AP}$ and for all $s, s' \in S$, $\ell'(s, s') = (\ell(s), \ell(s'))$.

A coupling model can be seen as a probabilistic pairing of two copies of \mathcal{M} running synchronously. The runs generated by a coupling model are *coupled timed paths* of the form $(s_0, s'_0), (t_0, t'_0), (s_1, s'_1), (t_1, t'_1), \dots$, i.e., infinite alternating sequences over $(S \times S)$ and $(\mathbb{R}_+ \times \mathbb{R}_+)$. For the sake of readability, we denote by $\langle \pi, \pi' \rangle$ a coupled timed path pairing $\pi = s_0, t_0, s_1, t_1, \dots$ and $\pi' = s'_0, t'_0, s'_1, t'_1, \dots$; and by $\Pi^2(S)$ the set of coupled timed paths over S .

The measurable space $\Pi^2(S, 2^S)$ of coupled timed paths over S has $\Pi^2(S)$ as underlying set and σ -algebra $\Sigma_{\Pi^2(S)} = \sigma(\mathfrak{C}(2^{S \times S}, \Sigma_{\mathbb{R}_+} \otimes \Sigma_{\mathbb{R}_+}))$. Couplings induce probability measures over the space of coupled timed paths as follows.

Definition 29. Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM, $\mathcal{C} = (S', A', \tau', \rho', \ell')$ be a coupling for it, and $s' \in S'$. The probability measure $\mathbb{P}_{s'}^{\mathcal{C}}$ on the measurable space of coupled timed paths over S is the unique measure such that, for all $n \in \mathbb{N}$, $s'_i \in S'$ and $R'_i \in \Sigma_{\mathbb{R}_+} \otimes \Sigma_{\mathbb{R}_+}$, $i = 0..n+1$

$$\begin{aligned} \mathbb{P}_{s'}^{\mathcal{C}}(\mathfrak{C}(s'_0)) &= \chi_{\{s'\}}(s'_0), \\ \mathbb{P}_{s'}^{\mathcal{C}}(\mathfrak{C}(s'_0, R'_0, \dots, s'_n, R'_n, s'_{n+1})) &= \mathbb{P}_{s'}^{\mathcal{C}}(\mathfrak{C}(s'_0, R'_0, \dots, s'_n)) \cdot P'(s'_n, R'_n, s'_{n+1}), \end{aligned}$$

where, for $u', v' \in S'$ and $R' \in \Sigma_{\mathbb{R}_+} \otimes \Sigma_{\mathbb{R}_+}$, $P'(u', R', v') = \rho'(u')(R') \cdot \tau'(u')(v')$, if $u' \notin A'$, and $P(u', R', v') = 0$ otherwise.

The existence and the uniqueness of this measure follows from Hahn-Kolmogorov extension theorem [23].

The name “coupling model” is justified by the fact that any measure $\mathbb{P}_{s, s'}^{\mathcal{C}}$ corresponds to a coupling for $(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}})$. This correspondence is obtained as a push forward w.r.t. the measurable function $\eta : \Pi^2(S) \rightarrow \Pi(S) \times \Pi(S)$ that assigns to a coupled timed path $\langle \pi, \pi' \rangle$ its (decoupled) pair (π, π') .

Lemma 30. Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM, \mathcal{C} an arbitrary coupling model for \mathcal{M} and $s, s' \in S$, then $\mathbb{P}_{s, s'}^{\mathcal{C}} \# \eta \in \Omega(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}})$.

Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM, we define $\vartheta^{\mathcal{M}} : S \times S \rightarrow [0, 1]$ as

$$\vartheta^{\mathcal{M}}(s, s') = \min \{ (\mathbb{P}_{s, s'}^{\mathcal{C}} \# \eta)(\neq_{\ell}^{\omega}) \mid \mathcal{C} \text{ coupling for } \mathcal{M} \}. \quad (1)$$

By Lemmas 30 and 27 we obtain that $\vartheta^{\mathcal{M}}$ is an over-approximation of $\delta^{\mathcal{M}}$.

Theorem 31. Let s, s' be states of the SMM \mathcal{M} . Then, $\delta^{\mathcal{M}}(s, s') \leq \vartheta^{\mathcal{M}}(s, s')$.

Fixed point characterization. In this section we give a fixed point characterization of $\vartheta^{\mathcal{M}}$ based on the Kantorovich metric and the total variation distance.

Given a coupling model \mathcal{C} for the SMM \mathcal{M} , and a pair of states s, s' of \mathcal{M} , the discrepancy between s and s' on \mathcal{C} , namely $(\mathbb{P}_{s,s'}^{\mathcal{C}} \# \eta)(\neq_{\ell}^{\omega})$, can be interpreted as a reachability probability. Indeed, for any pair of timed paths $\pi, \pi' \in \Pi(S)$, that starts in s and s' respectively, the probability associated with the event $\pi \neq_{\ell}^{\omega} \pi'$ corresponds to the probability of reaching a point $k \in \mathbb{N}$, where $\pi[k] \neq \pi'[k]$ or $\pi\langle k \rangle \neq \pi'\langle k \rangle$. In the spirit of the fixed point characterization of reachability probabilities in the case of MCs [8, Theorem 10.15], we give a fixed point characterization of the discrepancy between states.

Consider the set of $[0, 1]$ -valued functions on $S \times S$, denoted by $[0, 1]^{S \times S}$, endowed with the partial order \sqsubseteq defined by $d \sqsubseteq d'$ iff $d(s, s') \leq d'(s, s')$ for all $s, s' \in S$. This forms a complete lattice with bottom $\mathbf{0}$ and top $\mathbf{1}$, defined as $\mathbf{0}(s, s') = 0$ and $\mathbf{1}(s, s') = 1$, for all $s, s' \in S$. For $D \subseteq [0, 1]^{S \times S}$, the least upper bound $\bigsqcup D$, and greatest lower bound $\bigsqcap D$ are, respectively, given as $(\bigsqcup D)(s, t) = \sup_{d \in D} d(s, t)$ and $(\bigsqcap D)(s, t) = \inf_{d \in D} d(s, t)$, for all $s, t \in S$.

Consider an SMM $\mathcal{M} = (S, A, \tau, \rho, \ell)$ and a coupling $\mathcal{C} = (S', A', \tau', \rho', \ell')$ for \mathcal{M} . We define $\Gamma^{\mathcal{C}}: [0, 1]^{S \times S} \rightarrow [0, 1]^{S \times S}$ for $d: S \times S \rightarrow [0, 1]$ by

$$\Gamma^{\mathcal{C}}(d)(s, s') = \begin{cases} 1 & \text{if } s \not\equiv s' \\ 0 & \text{if } s \equiv s', \text{ and } s, s' \in A \\ \alpha + (1 - \alpha) \sum_{u, v \in S} d(u, v) \tau'(s, s')(u, v) & \text{otherwise} \end{cases} \quad \text{for } \alpha = \rho'(s, s')(\neq_{\mathbb{R}_+})$$

where $\neq_{\mathbb{R}_+}$ denotes the measurable set $\{(x, y) \in \mathbb{R}_+^2 \mid x \neq y\} \in \Sigma_{\mathbb{R}_+} \otimes \Sigma_{\mathbb{R}_+}$.

One can easily verify that $\Gamma^{\mathcal{C}}$ is well-defined and order preserving, so that, by Tarski's fixed point theorem, $\Gamma^{\mathcal{C}}$ admits a least fixed point, denoted by $\gamma^{\mathcal{C}}$.

The lemma below states that $\gamma^{\mathcal{C}}$ corresponds to the discrepancy w.r.t. \mathcal{C} .

Lemma 32. *Let \mathcal{C} be a coupling for an SMM $\mathcal{M} = (S, A, \tau, \rho, \ell)$, and $s, s' \in S$. Then, $(\mathbb{P}_{s,s'}^{\mathcal{C}} \# \eta)(\neq_{\ell}^{\omega}) = \gamma^{\mathcal{C}}(s, s')$.*

As an immediate corollary we obtain the following characterization of $\vartheta^{\mathcal{M}}$.

Corollary 33. *Let \mathcal{M} be an SMM. Then, $\vartheta^{\mathcal{M}} = \min \{\gamma^{\mathcal{C}} \mid \mathcal{C} \text{ coupling for } \mathcal{M}\}$.*

In [10,7] it is given a coupling-based characterization of the bisimilarity distance of Desharnais et al. [15] for MCs, starting from the fixed point characterization by van Breugel et al. [28]. Here we follow the same idea but backwards: we start from Corollary 33 to give a fixed point characterization of $\vartheta^{\mathcal{M}}$.

The following fixed point operator will do the job.

For $\mathcal{M} = (S, A, \tau, \rho, \ell)$ a SMM, the operator $F^{\mathcal{M}}: [0, 1]^{S \times S} \rightarrow [0, 1]^{S \times S}$, for $d: S \times S \rightarrow [0, 1]$ and $s, s' \in S$, is defined by:

$$F^{\mathcal{M}}(d)(s, s') = \begin{cases} 1 & \text{if } s \not\equiv s' \\ 0 & \text{if } s \equiv s' \text{ and } s, s' \in A \\ \alpha + (1 - \alpha) \mathcal{K}_d(\tau(s), \tau(s')) & \text{otherwise} \end{cases} \quad \text{for } \alpha = \|\rho(s) - \rho(s')\|_{\text{TV}}$$

where, for arbitrary $d: S \times S \rightarrow [0, 1]$ and $\mu, \nu \in \mathcal{D}(S)$, \mathcal{K}_d is the Kantorovich pseudometric that equals to $\mathcal{K}_d(\mu, \nu) = \min_{\omega \in \Omega(\mu, \nu)} \sum_{u, v \in S} \omega(u, v) \cdot d(u, v)$ in the finite discrete case⁸.

$F^{\mathcal{M}}$ is easily seen to be monotonic, thus, by Tarski's fixed point theorem, it has a least fixed point, that we denote by $f^{\mathcal{M}}$.

The next lemma states that the discrepancy measured w.r.t. arbitrary couplings \mathcal{C} for \mathcal{M} is an over-approximation of $f^{\mathcal{M}}$.

Lemma 34. *Let \mathcal{C} be a coupling for an SMM \mathcal{M} . If $d = \Gamma^{\mathcal{C}}(d)$, then $f^{\mathcal{M}} \subseteq d$.*

Lemma 34 and Corollary 33 yield the following characterization for $\vartheta^{\mathcal{M}}$.

Lemma 35. *Let \mathcal{M} be an SMM. Then, $f^{\mathcal{M}} = \vartheta^{\mathcal{M}}$.*

By Lemma 35 we can finally prove that $\vartheta^{\mathcal{M}}$ is a pseudometric for SMMs. Furthermore, we show that $\vartheta^{\mathcal{M}}$ is a bisimilarity pseudometric in the sense of [17].

Theorem 36. *Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM.*

1. $\vartheta^{\mathcal{M}}$ is a pseudometric;
2. for all $s, s' \in S$, $\vartheta^{\mathcal{M}}(s, s') = 0$ if and only if $s \sim_{\mathcal{M}} s'$.

6.2 Complexity Results

Provided that one has a way to compute the total variation distance between the residence-time probability distributions, Corollary 33 allows one to apply the same idea of [7] to obtain an on-the-fly algorithm for computing $\vartheta^{\mathcal{M}}$. This is the case for a relevant subclass of SMMs that includes MCs, CTMCs, and Markov models with uniformly or normally distributed residence time on states. For instance, the total variation distance between two exponential distributions with parameters $\lambda, \lambda' > 0$ is given by

$$\|Exp(\lambda) - Exp(\lambda')\|_{TV} = \begin{cases} 0 & \text{if } \lambda = \lambda' \\ \left| e^{\frac{\lambda'(\log \lambda - \log \lambda')}{\lambda' - \lambda}} - e^{\frac{\lambda(\log \lambda + \log \lambda')}{\lambda - \lambda'}} \right| & \text{otherwise} \end{cases}$$

Moreover, by Lemma 35 we can also establish good theoretical results.

From a theoretical point of view, it is irrelevant whether the transition probabilities have rational values or not. However, for the complexity results that follow we assume that, for all $s \in S \setminus A$ and $s' \in S$, it holds $\tau(s)(s') \in \mathbb{Q} \cap [0, 1]$.

Lemma 37. *Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM. Then, $\sim_{\mathcal{M}}$ can be computed in polynomial time in $\text{size}(\mathcal{M})$, provided that, for all $s, s' \in S \setminus A$, the value $\|\rho(s) - \rho(s')\|_{TV}$ can be computed in polynomial time in $\text{size}(\mathcal{M})$.*

The proof of Theorem 38 is based on the same idea of [10]. Specifically, $\vartheta^{\mathcal{M}}$ is characterized as the solution of a linear program that can be solved in polynomial time using the ellipsoid method [26].

⁸ Since S is finite, $\Omega(\mu, \nu)$ describes a *bounded* transportation polytope, hence the minimum in the definition of $\mathcal{K}_d(\mu, \nu)$ exists and can be achieved at some vertex.

Theorem 38. *Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM. Then, $\vartheta^{\mathcal{M}}$ can be computed in polynomial time in $\text{size}(\mathcal{M})$, provided that, for all $s, s' \in S \setminus A$, the value $\|\rho(s) - \rho(s')\|_{TV}$ can be computed in polynomial time in $\text{size}(\mathcal{M})$.*

7 Conclusions and Future Work

In this paper we took a step forward supporting approximate reasoning on Stochastic Markov Models in particular with respect to quantitative verification against linear real-time specifications expressed as MTL formulas or DTAs.

(i) We showed that the real-time specifications expressed as MTL formulas or DTAs generate the same σ -algebra. Then using a suitable pseudometric between specifications, we establish the subclass of resetting single-clock DTAs to be dense in the whole σ -algebra. (ii) We propose two behavioral distances between SMMs, defined as the maximal variation w.r.t. the probability of attaining respectively any MTL or DTA specification, and we showed that the two coincide to the total variation measured on the above mentioned σ -algebra. (iii) We established the **NP**-hardness of computing these distances exactly or within a certain absolute error that depends on the size of the system. (iv) Nevertheless, we defined a Kantorovich-based bisimilarity distance that over approximates these variation distances and can be computed in polynomial time.

Each of the previous points are supported by practical motivations.

As future work we will investigate other possible logical characterization, e.g. considering CSL or MTL with continuous semantics. From the computational perspective, also motivated by our recent work [7,6] on MCs and MDPs, we would like to implement an on-the-fly algorithm for computing $\vartheta^{\mathcal{M}}$ and develop a compositional theory for these models. To the best of our knowledge, whether the distance $\delta^{\mathcal{M}}$ is computable or not still remains an open problem.

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A Technical proofs

This section contains all the technical proofs that have been omitted in the paper and, in addition, some technical lemmas that are required in the technical development but are not the main exposition.

Proof (of Lemma 5). Immediate by Lemma 8 and $\sigma(\mathcal{T}_{\mathcal{M}}) \subseteq \Sigma_{\Pi(S, 2^S)}$. \square

Proof (of Lemma 7). Immediate by Lemma 8 and $\sigma(\mathcal{T}_{\mathcal{M}}) \subseteq \Sigma_{\Pi(S, 2^S)}$. \square

Lemma 39. *Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM and $tl_{\mathcal{M}}: \Pi(S) \rightarrow \Pi(S)$ be defined as $tl_{\mathcal{M}}(\pi) = \pi|_1$. Then, $tl_{\mathcal{M}}$ is measurable w.r.t $(\Pi(S), \sigma(\mathcal{T}_{\mathcal{M}}))$.*

Proof. It suffices to show that, for any $C \in \mathcal{T}_{\mathcal{M}}$, it holds $tl_{\mathcal{M}}^{-1}(C) \in \sigma(\mathcal{T}_{\mathcal{M}})$.

$$\begin{aligned} tl_{\mathcal{M}}^{-1}(C) &= \{\pi \in \Pi(S) \mid tl_{\mathcal{M}}(\pi) \in C\} && \text{(def. pre-image)} \\ &= \{\pi \in \Pi(S) \mid \pi|_1 \in C\}. && \text{(def. } tl_{\mathcal{M}}) \end{aligned}$$

by definition of $\mathcal{T}_{\mathcal{M}}$, $C = \mathfrak{C}([s_0]_{\equiv_{\ell}}, R_0, \dots, R_{n-1}, [s_n]_{\equiv_{\ell}})$ for some $n \in \mathbb{N}$, and some $s_i \in S$ and $R_i \in \Sigma_{\mathbb{R}_+}$ ($i = 0..n$)

$$\begin{aligned} &= \mathfrak{C}(S, \mathbb{R}_+, [s_0]_{\equiv_{\ell}}, R_0, \dots, R_{n-1}, [s_n]_{\equiv_{\ell}}) \\ &= \bigcup_{s \in S} \mathfrak{C}([s]_{\equiv_{\ell}}, \mathbb{R}_+, [s_0]_{\equiv_{\ell}}, R_0, \dots, R_{n-1}, [s_n]_{\equiv_{\ell}}) \end{aligned}$$

hence $tl_{\mathcal{M}}^{-1}(C) \in \sigma(\mathcal{T}_{\mathcal{M}})$ since it is a (finite) union of cylinders in $\mathcal{T}_{\mathcal{M}}$ \square

Proof (of Lemma 8).

Proof of $\sigma(\llbracket \text{MTL} \rrbracket_{\mathcal{M}}) = \sigma(\mathcal{T}_{\mathcal{M}})$:

(\subseteq) It suffices to prove that $\llbracket \text{MTL} \rrbracket_{\mathcal{M}} \subseteq \sigma(\mathcal{T}_{\mathcal{M}})$. We proceed by structural induction on the formulas $\varphi \in \text{MTL}$ showing that $\llbracket \varphi \rrbracket_{\mathcal{M}} \in \sigma(\mathcal{T}_{\mathcal{M}})$.

Atomic prop. $\llbracket a \rrbracket_{\mathcal{M}} = \{\pi \mid a \in \ell(\pi[0])\} = \bigcup \{\mathfrak{C}([s]_{\equiv_{\ell}}) \mid s \in \ell^{-1}(\{a\})\}$. Since S is finite and $\mathfrak{C}([s]_{\equiv_{\ell}}) \in \mathcal{T}_{\mathcal{M}}$ for all $s \in S$, then $\llbracket a \rrbracket_{\mathcal{M}} \in \sigma(\mathcal{T}_{\mathcal{M}})$.

False. $\llbracket \perp \rrbracket_{\mathcal{M}} = \emptyset \in \sigma(\mathcal{T}_{\mathcal{M}})$.

Implication. $\llbracket \phi \rightarrow \psi \rrbracket_{\mathcal{M}} = \llbracket \neg \phi \vee \psi \rrbracket_{\mathcal{M}} = \llbracket \phi \rrbracket_{\mathcal{M}}^c \cup \llbracket \psi \rrbracket_{\mathcal{M}}$. By inductive hypothesis, $\llbracket \phi \rrbracket_{\mathcal{M}}, \llbracket \psi \rrbracket_{\mathcal{M}} \in \sigma(\mathcal{T}_{\mathcal{M}})$, therefore $\llbracket \phi \rightarrow \psi \rrbracket_{\mathcal{M}} \in \Sigma_{\mathcal{M}}$.

Next. Consider $\mathbf{X}^I \phi$. The following hold

$$\begin{aligned} \llbracket \mathbf{X}^I \phi \rrbracket_{\mathcal{M}} &= \{\pi \mid \pi\langle 0 \rangle \in I, \text{ and } \mathcal{M}, \pi|_1 \models \phi\} && \text{(by def. of } \mathbf{X}) \\ &= \{\pi \mid \pi\langle 0 \rangle \in I, \text{ and } \pi|_1 \in \llbracket \phi \rrbracket_{\mathcal{M}}\} && \text{(by def. of } \llbracket \cdot \rrbracket_{\mathcal{M}}) \\ &= \{\pi \mid \pi|_1 \in \llbracket \phi \rrbracket_{\mathcal{M}}\} \cap \mathfrak{C}(S, I, S) \\ &= tl_{\mathcal{M}}^{-1}(\llbracket \phi \rrbracket_{\mathcal{M}}) \cap \mathfrak{C}(S, I, S). && \text{(by def. } tl_{\mathcal{M}}) \end{aligned}$$

$\mathfrak{C}(S, I, S) = \bigcup_{s, s' \in S} \mathfrak{C}([s]_{\equiv_{\ell}}, I, [s']_{\equiv_{\ell}}) \in \sigma(\mathcal{T}_{\mathcal{M}})$ since $\mathfrak{C}([s]_{\equiv_{\ell}}, I, [s']_{\equiv_{\ell}}) \in \mathcal{T}_{\mathcal{M}}$ for all $s, s' \in S$ and S is finite. By inductive hypothesis $\llbracket \phi \rrbracket_{\mathcal{M}} \in \sigma(\mathcal{T}_{\mathcal{M}})$, hence, by Lemma 39, $tl_{\mathcal{M}}^{-1}(\llbracket \phi \rrbracket_{\mathcal{M}}) \in \sigma(\mathcal{T}_{\mathcal{M}})$. Therefore $\llbracket \mathbf{X}^I \phi \rrbracket_{\mathcal{M}} \in \sigma(\mathcal{T}_{\mathcal{M}})$.

Until. Consider $\llbracket \phi \cup^{[a,b]} \psi \rrbracket_{\mathcal{M}}$. For $k > 0$ we define the set $OnTime@k$ as

$$OnTime@k = \bigcup \left\{ \mathfrak{C}(X) \left| \begin{array}{l} s_i \in S, t_i^-, t_i^+ \in \mathbb{Q}_+, \text{ for } 0 \leq i \leq k, \\ \sum_{i=0}^{k-1} t_i^- \geq a, \sum_{i=0}^{k-1} t_i^+ \leq b, t_i^- \leq t_i^+, \\ X = [s_0]_{\equiv_\ell}, [t_0^-, t_0^+], \dots, [t_{k-1}^-, t_{k-1}^+], [s_n]_{\equiv_\ell} \end{array} \right. \right\}$$

Notice that $OnTime@k$ is a countable union of cylinders in $\mathcal{T}_{\mathcal{M}}$ (the number of unions is bounded by $|(S \times \mathbb{Q}_+^2)^{k+1}|$), hence it is a measurable set in $\sigma(\mathcal{T}_{\mathcal{M}})$.

Now we show that

$$OnTime@k = \{ \pi \mid \forall i < k. \sum_{i=0}^{k-1} \pi\langle i \rangle \in [a, b] \} \quad (2)$$

The inclusion from left to right trivially holds by definition of $OnTime@k$. As for the reverse inclusion, let π be a timed path over S , such that $\pi\langle i \rangle = t_i$ ($i = 0..k-1$) and $\sum_{i=0}^{k-1} t_i \in [a, b]$. We have to prove that there exist $t_i^-, t_i^+ \in \mathbb{Q}_+$ such that $t_i^- \leq t_i \leq t_i^+$, $\sum_{i=0}^{k-1} t_i^- \geq a$, and $\sum_{i=0}^{k-1} t_i^+ \leq b$. When $k = 1$ it suffices to take $t_1^- = a$ and $t_1^+ = b$. Assume $k > 1$. Let $\Delta = 2h/10^h$ for some $h \in \mathbb{N}$ large enough to satisfy the following two inequalities $\sum_{i=0}^{k-1} t_i - \Delta > a$ and $\sum_{i=0}^{k-2} t_i + \Delta < b$. Let $t_i^- = t_i = t_i^+$ if $t_i \in \mathbb{Q}_+$, otherwise we choose some $t_i^-, t_i^+ \in \mathbb{Q}_+$ that satisfy

$$t_i^- < t_i < t_i^+, \quad t_i^- > t_i - \Delta/2k, \text{ and} \quad t_i^+ < t_i + \Delta/2k. \quad (3)$$

We proceed by showing that the constraints (3) are sufficient to prove that $\sum_{i=0}^{k-1} t_i^- \geq a$ and $\sum_{i=0}^{k-1} t_i^+ \leq b$, then we show how to pick such $t_i^-, t_i^+ \in \mathbb{Q}_+$ in order to satisfy (3). The following hold

$$\begin{aligned} \sum_{i=0}^{k-1} t_i - \Delta &< \sum_{i=0}^{k-1} (t_i^- + \Delta/2k) - \Delta = & (\text{by (3)}) \\ &= \sum_{i=0}^{k-1} t_i^- - \Delta/2 \leq \sum_{i=0}^{k-1} t_i^- & (\text{by } \Delta \geq 0) \end{aligned}$$

By construction $\sum_{i=0}^{k-1} t_i - \Delta > a$, hence $\sum_{i=0}^{k-1} t_i^- > a$. Analogously, we have that

$$\begin{aligned} \sum_{i=0}^{k-2} t_i + \Delta &> \sum_{i=0}^{k-1} (t_i^+ - \Delta/2k) - \Delta = & (\text{by (3)}) \\ &= \sum_{i=0}^{k-2} t_i^+ + (k+1)\Delta/2k \geq \sum_{i=0}^{k-2} t_i^+ & (\text{by } \Delta \geq 0) \end{aligned}$$

By construction $\sum_{i=0}^{k-2} t_i + \Delta < b$, hence $\sum_{i=0}^{k-2} t_i^+ < b$.

One can check that the constraints (3) are easily satisfied if we pick

$$t_i^- = \lfloor t_i \rfloor + \frac{\lfloor 10^h \cdot \{t_i\} \rfloor}{10^h}, \quad t_i^+ = \lfloor t_i \rfloor + \frac{\lfloor 10^h \cdot \{t_i\} \rfloor + 1}{10^h},$$

where $\{t_i\}$ denotes the fractional part of $t_i \notin \mathbb{Q}_+$. This proves (2).

$$\begin{aligned}
& \llbracket \phi \mathbf{U}^{[a,b]} \psi \rrbracket_{\mathcal{M}} \\
&= \left\{ \pi \left| \begin{array}{l} \exists i > 0. \sum_{k=0}^{i-1} \pi \langle i \rangle \in [a, b], \text{ and } \mathcal{M}, \pi|_i \models \psi, \\ \forall 0 \leq j < i. \mathcal{M}, \pi|_j \models \phi \end{array} \right. \right\} \quad (\text{by def. } \mathbf{U}) \\
&= \left\{ \pi \left| \begin{array}{l} \exists i > 0. \sum_{k=0}^{i-1} \pi \langle i \rangle \in [a, b], \text{ and } \pi|_i \in \llbracket \psi \rrbracket_{\mathcal{M}}, \\ \forall 0 \leq j < i. \pi|_j \in \llbracket \phi \rrbracket_{\mathcal{M}} \end{array} \right. \right\} \quad (\text{by def. } \llbracket \cdot \rrbracket_{\mathcal{M}}) \\
&= \bigcup_{i>0} \bigcap_{0 \leq j < i} ((tl_{\mathcal{M}}^j)^{-1}(\llbracket \phi \rrbracket_{\mathcal{M}}) \cap (tl_{\mathcal{M}}^i)^{-1}(\llbracket \psi \rrbracket_{\mathcal{M}}) \cap \text{OnTime}@i). \\
& \hspace{15em} (\text{by def. } tl_{\mathcal{M}} \text{ and (2)})
\end{aligned}$$

By inductive hypothesis $\llbracket \phi \rrbracket_{\mathcal{M}}, \llbracket \psi \rrbracket_{\mathcal{M}} \in \sigma(\mathcal{T}_{\mathcal{M}})$, so that, by Lemma 39 and since $\sigma(\mathcal{T}_{\mathcal{M}})$ is closed under countable union and intersection, $\llbracket \phi \mathbf{U}^{[a,b]} \psi \rrbracket_{\mathcal{M}} \in \sigma(\mathcal{T}_{\mathcal{M}})$.

(\supseteq) Let \mathcal{I} be the family of closed intervals in \mathbb{R}_+ with rational endpoints. Clearly, $\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I}) \subseteq \mathcal{T}_{\mathcal{M}}$, hence $\sigma(\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I})) \subseteq \sigma(\mathcal{T}_{\mathcal{M}})$. It is standard that $\sigma(\mathcal{I}) = \Sigma_{\mathbb{R}_+}$, and from it one can easily verify that $\mathcal{T}_{\mathcal{M}} \subseteq \sigma(\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I}))$, therefore we have also $\sigma(\mathcal{T}_{\mathcal{M}}) \subseteq \sigma(\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I}))$. This proves $\sigma(\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I})) = \sigma(\mathcal{T}_{\mathcal{M}})$.

From this equality, to show the inclusion $\sigma(\mathcal{T}_{\mathcal{M}}) \subseteq \sigma(\llbracket \text{MTL} \rrbracket_{\mathcal{M}})$, it suffices to prove that $\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I}) \subseteq \sigma(\llbracket \text{MTL} \rrbracket_{\mathcal{M}})$.

Let define $Ap: AP \times \mathfrak{C}(S/\equiv_{\ell}, \mathcal{I}) \rightarrow \text{MTL}$ as

$$\begin{aligned}
Ap(a, \mathfrak{C}([s]_{\equiv_{\ell}})) &= \begin{cases} a & \text{if } a \in \ell(s) \\ \neg a & \text{otherwise} \end{cases} \\
Ap(a, \mathfrak{C}([s]_{\equiv_{\ell}}, I, X)) &= \begin{cases} a \wedge \mathbf{X}^I Ap(a, \mathfrak{C}(X)) & \text{if } a \in \ell(s) \\ \neg a \wedge \mathbf{X}^I Ap(a, \mathfrak{C}(X)) & \text{otherwise,} \end{cases}
\end{aligned}$$

Let $C = \mathfrak{C}([s_0]_{\equiv_{\ell}}, I_0, \dots, I_{n-1}, [s_n]_{\equiv_{\ell}}) \in \mathfrak{C}(S/\equiv_{\ell}, \mathcal{I})$, one can prove by induction on n that $\bigcap_{a \in AP} \llbracket Ap(a, C) \rrbracket_{\mathcal{M}} = C$. Since $\sigma(\llbracket \text{MTL} \rrbracket_{\mathcal{M}})$ is closed under countable intersection, we conclude that $C \in \sigma(\llbracket \text{MTL} \rrbracket_{\mathcal{M}})$.

Proof of $\sigma(\llbracket \text{DTA} \rrbracket_{\mathcal{M}}) = \sigma(\mathcal{T}_{\mathcal{M}})$:

(\subseteq) It suffices to show that $\llbracket \text{DTA} \rrbracket_{\mathcal{M}} \subseteq \sigma(\mathcal{T}_{\mathcal{M}})$. This is proven in [12, Theorem 3.2] and the proof can be left unchanged.

(\supseteq) Let \mathcal{I} be the family of closed intervals in \mathbb{R}_+ with rational endpoints. Clearly, $\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I}) \subseteq \mathcal{T}_{\mathcal{M}}$, hence $\sigma(\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I})) \subseteq \sigma(\mathcal{T}_{\mathcal{M}})$. It is standard that $\sigma(\mathcal{I}) = \Sigma_{\mathbb{R}_+}$, and from it one can easily verify that $\mathcal{T}_{\mathcal{M}} \subseteq \sigma(\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I}))$, therefore we have also $\Sigma_{\mathcal{M}} \subseteq \sigma(\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I}))$. This proves $\sigma(\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I})) = \sigma(\mathcal{T}_{\mathcal{M}})$.

Hence, for $\Sigma_{\mathcal{M}} \subseteq \sigma(\llbracket \text{DTA} \rrbracket_{\mathcal{M}})$ it suffices to prove $\mathfrak{C}(S/\equiv_{\ell}, \mathcal{I}) \subseteq \sigma(\llbracket \text{DTA} \rrbracket_{\mathcal{M}})$. Let $C = \mathfrak{C}([s_0]_{\equiv_{\ell}}, I_0, \dots, I_{n-1}, [s_n]_{\equiv_{\ell}}) \in \mathfrak{C}(S/\equiv_{\ell}, \mathcal{I})$. We want to define a DTA $\mathcal{A} = (Q, 2^{AP}, q_0, F, \rightarrow)$ such that $\llbracket \mathcal{A} \rrbracket_{\mathcal{M}} = C$. This can be obtained by setting $Q = \{q_0, \dots, q_n\}$, $F = \{q_n\}$, and, using one shared clock $x \in \mathcal{X}$ in each guard,

$$\begin{aligned}
\rightarrow &= \{(q_i, \ell(s_i), g_i, \mathcal{X}, q_{i+1}) \mid g_i = a \leq x \leq b \text{ for } I_i = [a, b], 0 \leq i \leq n\} \\
&\cup \{(q_n, a, \emptyset, \mathcal{X}, q_n) \mid a \subseteq AP\}.
\end{aligned}$$

It is easy to see that the only accepted timed paths $\pi \in \mathcal{L}(\mathcal{A})$ are such that $\pi|_n = \ell(s_0), t_0, \dots, t_{n-1}, \ell(s_n)$, and $t_i \in I_i$ ($0 \leq i \leq n-1$), because clocks are always resetting. So the thesis. \square

Proof (of Lemma 9). We have to show that $\overline{\mathcal{F}} = \Sigma$. The closure of \mathcal{F} under d_μ is given by

$$\overline{\mathcal{F}} = \{E \in \Sigma \mid \forall \varepsilon > 0. \exists F \in \mathcal{F}. d_\mu(E, F) < \varepsilon\}.$$

Clearly $\overline{\mathcal{F}} \subseteq \Sigma$. The converse inclusion follows by $\mathcal{F} \subseteq \overline{\mathcal{F}}$ and $\Sigma = \sigma(\mathcal{F})$, showing that $\overline{\mathcal{F}}$ is a σ -algebra, i.e., closed under complement and countable union.

Complement. Let $E \in \overline{\mathcal{F}}$. We want to show that $E^c \in \overline{\mathcal{F}}$, where $E^c := \Sigma \setminus E$ denotes the complement of E in Σ . Let $\varepsilon > 0$. By $E \in \overline{\mathcal{F}}$, there exists $F \in \mathcal{F}$ such that $d_\mu(E, F) < \varepsilon$. Moreover, note that $E \triangle F = E^c \triangle F^c$, so

$$d_\mu(E^c, F^c) = \mu(E^c \triangle F^c) = \mu(E \triangle F) = d_\mu(E, F),$$

and $d_\mu(E^c, F^c) < \varepsilon$. By hypothesis, \mathcal{F} is a field, hence $F^c \in \mathcal{F}$. Due to the generality of $\varepsilon > 0$, this proves $E^c \in \overline{\mathcal{F}}$.

Countable Union. Let $\{E_i \mid i \in \mathbb{N}\} \subseteq \overline{\mathcal{F}}$. We want to show that $\bigcup_{i \in \mathbb{N}} E_i \in \overline{\mathcal{F}}$. Let $\varepsilon > 0$. To prove the thesis it suffices to show that following statements hold:

- a) there exists $k \in \mathbb{N}$, such that $d_\mu(\bigcup_{i \in \mathbb{N}} E_i, \bigcup_{i=0}^k E_i) < \frac{\varepsilon}{2}$;
 - b) for all $n \in \mathbb{N}$, there exist $F_0, \dots, F_n \in \mathcal{F}$, such that $d_\mu(\bigcup_{i=0}^n E_i, \bigcup_{i=0}^n F_i) < \frac{\varepsilon}{2}$.
- Indeed, by applying the triangular inequality on (a) and (b), we have that there exist $k \in \mathbb{N}$ and $F_0, \dots, F_k \in \mathcal{F}$ such that

$$d_\mu(\bigcup_{i \in \mathbb{N}} E_i, \bigcup_{i=0}^k F_i) \leq d_\mu(\bigcup_{i \in \mathbb{N}} E_i, \bigcup_{i=0}^k E_i) + d_\mu(\bigcup_{i=0}^k E_i, \bigcup_{i=0}^k F_i) < \varepsilon.$$

Since, by hypothesis, \mathcal{F} is a field, we also have that $\bigcup_{i=0}^k F_i \in \mathcal{F}$. Therefore, due to the generality of $\varepsilon > 0$, we will obtain that $\bigcup_{i \in \mathbb{N}} E_i \in \overline{\mathcal{F}}$.

(a). Since $(\bigcup_{i=0}^n E_i)_{n \in \mathbb{N}}$ is a countable increasing sequence in Σ converging to $\bigcup_{i \in \mathbb{N}} E_i$, by ω -continuity from below of μ , we have that $(\mu(\bigcup_{i=0}^n E_i))_{n \in \mathbb{N}}$ converges in \mathbb{R} to $\mu(\bigcup_{i \in \mathbb{N}} E_i)$. This means that there exists an index $k \in \mathbb{N}$ such that

$$|\mu(\bigcup_{i \in \mathbb{N}} E_i) - \mu(\bigcup_{i=0}^k E_i)| < \frac{\varepsilon}{2}$$

By $\bigcup_{i \in \mathbb{N}} E_i \subseteq \bigcup_{i=0}^k E_i$ and monotonicity, additivity, and finiteness of μ ,

$$\begin{aligned} d_\mu(\bigcup_{i \in \mathbb{N}} E_i, \bigcup_{i=0}^k E_i) &= \mu(\bigcup_{i \in \mathbb{N}} E_i \triangle \bigcup_{i=0}^k E_i) \\ &= \mu(\bigcup_{i \in \mathbb{N}} E_i \setminus \bigcup_{i=0}^k E_i) \\ &= \mu(\bigcup_{i \in \mathbb{N}} E_i) - \mu(\bigcup_{i=0}^k E_i) < \frac{\varepsilon}{2}. \end{aligned}$$

(b). Let $n \in \mathbb{N}$. By $E_0, \dots, E_n \in \overline{\mathcal{F}}$, there exists $F_0, \dots, F_n \in \mathcal{F}$ such that $d_\mu(E_i, F_i) < \frac{\varepsilon}{2n}$. Moreover, note that $\bigcup_{i=0}^n E_i \triangle \bigcup_{i=0}^n F_i \subseteq \bigcup_{i=0}^n (E_i \triangle F_i)$, so that by monotonicity and sub-additivity of μ we have

$$\begin{aligned} d_\mu(\bigcup_{i=0}^n E_i, \bigcup_{i=0}^n F_i) &= \mu(\bigcup_{i=0}^n E_i \triangle \bigcup_{i=0}^n F_i) \\ &\leq \mu(\bigcup_{i=0}^n (E_i \triangle F_i)) \\ &\leq \sum_{i=0}^n \mu(E_i \triangle F_i) < \sum_{i=0}^n \frac{\varepsilon}{2n} = \frac{\varepsilon}{2}. \end{aligned}$$

□

Proof (of Lemma 11). One inclusion follows obviously since $\llbracket 1\text{-RDTA} \rrbracket_{\mathcal{M}} \subseteq \llbracket \text{DTA} \rrbracket_{\mathcal{M}}$. The converse inclusion is already proven in Lemma 8 which uses single-clock resetting DTAs only. □

Proof (of Theorem 12). By Lemmas 8 and 11, $\llbracket 1\text{-RDTA} \rrbracket_{\mathcal{M}}$ generates $\Sigma_{\mathcal{M}}$, moreover it is a field. Therefore, by Lemma 9, $\llbracket 1\text{-RDTA} \rrbracket_{\mathcal{M}}$ is dense in $(\Sigma_{\mathcal{M}}, d_{\mu})$, for all finite measures μ over $(\Pi(S), \Sigma_{\mathcal{M}})$. To prove the thesis it suffices to show that the function $\mu: \Sigma \rightarrow \mathbb{R}$ is continuous. Let E and F be arbitrary measurable sets in Σ , then

$$\begin{aligned}
 \mu(E) &= \mu(E \setminus F) + \mu(E \cap F) && (\mu \text{ additive}) \\
 &\leq \mu((E \setminus F) \cup (F \setminus E)) + \mu(F) && (\mu \text{ monotone}) \\
 &= \mu(E \triangle F) + \mu(E) && (\text{by def}) \\
 &= \mu(E \triangle F) + \nu(E \triangle F) + \mu(F) && (\nu \text{ positive}) \\
 &= d_{\mu}(E, F) + \mu(F). && (\text{by def})
 \end{aligned}$$

This implies that, for all $E, F \in \Sigma$, $d(E, F) \geq |\mu(E) - \mu(F)|$, hence $\mu: \Sigma \rightarrow \mathbb{R}$ is 1-Lipschitz continuous. □

Proposition 40. *Let $A \subseteq \mathbb{R}$ be a bounded nonempty set. Then,*

- (i) $\sup A \in \overline{A}$;
- (ii) $\sup A = \sup \overline{A}$.

Proof. First, notice that since $A \neq \emptyset$ and is bounded, by Dedekind axiom, the supremum of A (and \overline{A}) in \mathbb{R} exists. Moreover, recall that, for any $B \subseteq \mathbb{R}$,

$$\overline{B} = ad(B) := \{x \in \mathbb{R} \mid \forall \varepsilon > 0. (x - \varepsilon, x + \varepsilon) \cap B \neq \emptyset\},$$

where $ad(B)$ denotes the set of points *adherent* to B .

Let $\alpha = \sup A$. (i) We prove that $\alpha \in \overline{A}$. Let $\varepsilon > 0$, then $\alpha - \varepsilon$ is not an upper bound for A . This means that there exists $x \in A$ such that $\alpha - \varepsilon < x \leq \alpha$ and, in particular, that $x \in (\alpha - \varepsilon, \alpha + \varepsilon) \cap A$. Therefore $\alpha \in \overline{A}$. (ii) Let $\beta = \sup \overline{A}$. By $A \subseteq \overline{A} = \overline{\overline{A}}$ and (i), we have $\alpha \leq \beta \in \overline{A}$. We prove that $\alpha = \beta$. Assume by contradiction that $\alpha \neq \beta$ and let $\varepsilon := \beta - \alpha$. Clearly $\varepsilon > 0$, so that, by $\beta \in \overline{A}$, we have that $(\beta - \varepsilon, \beta + \varepsilon) \cap A \neq \emptyset$. This means that there exists $x \in A$ such that $\alpha = \beta - \varepsilon < x$, in contradiction with the hypothesis that $\alpha = \sup A$. □

Proposition 41. *Let $f: X \rightarrow Y$ be continuous and $A \subseteq X$, then $\overline{f(A)} = \overline{f(\overline{A})}$.*

Proof. (\supseteq) A function $f: X \rightarrow Y$ is continuous iff for all $B \subseteq X$, $f(\overline{B}) \subseteq \overline{f(B)}$. Therefore $f(\overline{A}) \subseteq \overline{f(A)}$. Since $\overline{f(A)}$ is closed, we have $\overline{f(A)} \subseteq \overline{f(\overline{A})}$. (\subseteq) The result follows by $A \subseteq \overline{A}$ and monotonicity of $f(\cdot)$ and $\overline{(\cdot)}$. □

Proposition 42. *Let X be nonempty, $f: X \rightarrow \mathbb{R}$ be a bounded continuous real-valued function, and $D \subseteq X$ be dense in X . Then $\sup f(D) = \sup f(X)$.*

Proof. Notice that, since $X \neq \emptyset$ and f is bounded, by Dedekind axiom, both $\sup f(D)$ and $\sup f(X)$ exist. By Propositions 40, 41, and $\overline{D} = X$, we have

$$\sup f(D) \stackrel{(\text{Prop.40})}{=} \sup \overline{f(D)} \stackrel{(\text{Prop.41})}{=} \sup \overline{f(\overline{D})} = \sup \overline{f(X)} \stackrel{(\text{Prop.40})}{=} \sup f(X),$$

which proves the thesis. \square

Proof (of Lemma 14). Consider the pseudometric $d(E, F) = \mu(E \triangle F) + \nu(E \triangle F)$ on Σ , i.e., the Fréchet-Nikodym pseudometric w.r.t. $\mu + \nu$. By Lemma 9, we know that \mathcal{F} is dense in (Σ, d) . For any nonempty set Y and any bounded continuous real-valued function $f: Y \rightarrow \mathbb{R}$, if $D \subseteq Y$ is dense then $\sup f(D) = \sup f(X)$ (see Proposition 42). Therefore, to prove the thesis it suffices to show that the function $|\mu - \nu|: \Sigma \rightarrow \mathbb{R}$ is bounded and continuous. Let E and F be arbitrary measurable sets in Σ , then

$$\begin{aligned} \mu(E) &= \mu(E \setminus F) + \mu(E \cap F) && (\mu \text{ additive}) \\ &\leq \mu((E \setminus F) \cup (F \setminus E)) + \mu(F) && (\mu \text{ monotone}) \\ &= \mu(E \triangle F) + \mu(E) && (\text{by def}) \\ &= \mu(E \triangle F) + \nu(E \triangle F) + \mu(F) && (\nu \text{ positive}) \\ &= d(E, F) + \mu(F). && (\text{by def}) \end{aligned}$$

This implies that, for all $E, F \in \Sigma$, $d(E, F) \geq |\mu(E) - \mu(F)|$, hence $\mu: \Sigma \rightarrow \mathbb{R}$ is 1-Lipschitz continuous. Analogously, also $\nu: \Sigma \rightarrow \mathbb{R}$ is 1-Lipschitz continuous. Then, continuity of $|\mu - \nu|: \Sigma \rightarrow \mathbb{R}$ follows by composition of continuous functions. Moreover, $|\mu - \nu|$ is bounded since, by hypothesis, μ and ν are finite. \square

Proof (of Theorem 15). The thesis follows by Lemmas 14 and 8, noticing that $\llbracket \text{MTL} \rrbracket_{\mathcal{M}}$ and $\llbracket \text{DTA} \rrbracket_{\mathcal{M}}$ are fields. \square

Proof (of Theorem 16). (\supseteq) Immediate by $\mathcal{T}_{\mathcal{M}} \subseteq \Sigma_{\mathcal{M}}$. (\subseteq) It follows by Hahn-Kolmogorov extension theorem, by noticing that the family \mathcal{F} consisting of all finite unions of trace cylinders in $\mathfrak{C}(S/\equiv_{\ell}, \Sigma_{\mathbb{R}_+})$ is indeed a field, and since S/\equiv_{ℓ} has only pairwise disjoint subsets, the uniqueness of the extension w.r.t. \mathcal{F} implies the uniqueness of the extension w.r.t. $\mathfrak{C}(S/\equiv_{\ell}, \Sigma_{\mathbb{R}_+})$. \square

Proof (of Lemma 19). The thesis follows from the following equalities:

$$\sup_{E \in \Sigma_{\mathcal{M}}} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)| = \sup_{E \in \sigma(\mathcal{W}_{\mathcal{M}})} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)|, \quad (4)$$

$$2 \cdot \sup_{E \in \sigma(\mathcal{W}_{\mathcal{M}})} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)| = \sum_{E \in \mathcal{W}_{\mathcal{M}}} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)|. \quad (5)$$

Equation (4): (\geq) directly follows by $\sigma(\mathcal{W}_{\mathcal{M}}) \subseteq \sigma(\mathcal{T}_{\mathcal{M}}) = \Sigma_{\mathcal{M}}$.

(\leq) We prove by induction on $n \in \mathbb{N}$, that for any pair of cylinder sets C_n, C'_n of the form $\mathfrak{C}(s_0, R_0, \dots, R_{n-1}, s_n)$ and $\mathfrak{C}(s_0, \mathbb{R}_+, \dots, \mathbb{R}_+, s_n)$ respectively, it holds that $|\mathbb{P}_s^{\mathcal{M}}(C_n) - \mathbb{P}_{s'}^{\mathcal{M}}(C_n)| \leq |\mathbb{P}_s^{\mathcal{M}}(C'_n) - \mathbb{P}_{s'}^{\mathcal{M}}(C'_n)|$. The base case, $n = 0$,

holds trivially. For the inductive step, assume that $n \geq 1$ and, w.l.g. that $\mathbb{P}_s^{\mathcal{M}}(C_n) \geq \mathbb{P}_{s'}^{\mathcal{M}}(C_n)$. Then, the following hold

$$\begin{aligned}
& \mathbb{P}_s^{\mathcal{M}}(C_n) - \mathbb{P}_{s'}^{\mathcal{M}}(C_n) && \text{(by def of } \mathbb{P}^{\mathcal{M}}) \\
&= P(s_{n-1}, R_{n-1}, s_n) \cdot (\mathbb{P}_s^{\mathcal{M}}(C_{n-1}) - \mathbb{P}_{s'}^{\mathcal{M}}(C_{n-1})) && \text{(by monotonicity)} \\
&\geq P(s_{n-1}, \mathbb{R}_+, s_n) \cdot (\mathbb{P}_s^{\mathcal{M}}(C_{n-1}) - \mathbb{P}_{s'}^{\mathcal{M}}(C_{n-1})) && \text{(by ind. hp.)} \\
&\geq P(s_{n-1}, \mathbb{R}_+, s_n) \cdot (\mathbb{P}_s^{\mathcal{M}}(C'_{n-1}) - \mathbb{P}_{s'}^{\mathcal{M}}(C'_{n-1})) && \text{(by def of } \mathbb{P}^{\mathcal{M}}) \\
&= \mathbb{P}_s^{\mathcal{M}}(C'_n) - \mathbb{P}_{s'}^{\mathcal{M}}(C'_n)
\end{aligned}$$

By Hahn-Kolmogorow extension theorem and Hahn decomposition theorem on signed measures it follows that for all $E \in \Sigma_{\mathcal{M}}$ there exists $E' \in \sigma(\mathcal{W}_{\mathcal{M}})$ such that $|\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)| \leq |\mathbb{P}_s^{\mathcal{M}}(E') - \mathbb{P}_{s'}^{\mathcal{M}}(E')|$. This proves (4).

Equation (5): Since S is finite, $\mathcal{W}_{\mathcal{M}}$ has countably many elements, moreover, since \equiv_{ℓ} is an equivalence relation, they are also pairwise disjoint. Therefore, every measurable set $E \in \sigma(\mathcal{W}_{\mathcal{M}})$ can be expressed as a countable union of cylinders taken from $\mathcal{W}_{\mathcal{M}}$. Let \mathcal{P} be the family of all cylinders $C \in \mathcal{W}_{\mathcal{M}}$ such that $\mathbb{P}_s^{\mathcal{M}}(C) \geq \mathbb{P}_{s'}^{\mathcal{M}}(C)$. By Hahn decomposition theorem we have that $P = \bigcup \mathcal{P}$ is a positive set for the signed measure $\mathbb{P}_s^{\mathcal{M}} - \mathbb{P}_{s'}^{\mathcal{M}}$, that is

$$\sup_{E \in \sigma(\mathcal{W}_{\mathcal{M}})} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)| = \mathbb{P}_s^{\mathcal{M}}(P) - \mathbb{P}_{s'}^{\mathcal{M}}(P). \quad (6)$$

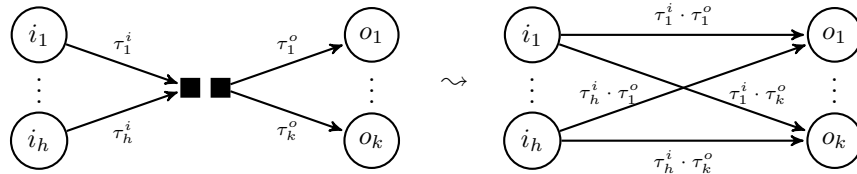
Now we are ready to prove (5)

$$\begin{aligned}
& 2 \cdot \sup_{E \in \sigma(\mathcal{W}_{\mathcal{M}})} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)| = && \text{(by (6))} \\
&= 2 \cdot (\mathbb{P}_s^{\mathcal{M}}(P) - \mathbb{P}_{s'}^{\mathcal{M}}(P)) = \\
&= \mathbb{P}_s^{\mathcal{M}}(P) - \mathbb{P}_{s'}^{\mathcal{M}}(P) + \mathbb{P}_{s'}^{\mathcal{M}}(P^c) - \mathbb{P}_s^{\mathcal{M}}(P^c) && (\sigma\text{-additivity}) \\
&= \sum_{C \in \mathcal{P}} (\mathbb{P}_s^{\mathcal{M}}(C) - \mathbb{P}_{s'}^{\mathcal{M}}(C)) + \sum_{C \notin \mathcal{P}} (\mathbb{P}_{s'}^{\mathcal{M}}(C) - \mathbb{P}_s^{\mathcal{M}}(C)) \\
&= \sum_{C \in \mathcal{W}_{\mathcal{M}}} |\mathbb{P}_s^{\mathcal{M}}(C) - \mathbb{P}_{s'}^{\mathcal{M}}(C)|.
\end{aligned}$$

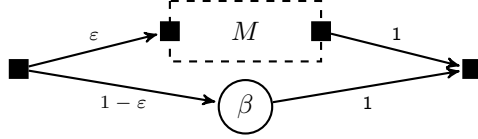
□

Proof (of Theorem 20). Let $\mathcal{G} = (V, E)$ be an undirected graph with n vertices, i.e., $n = |V|$, and let fix $\kappa \in \Delta(\mathbb{R}_+)$. For simplicity, we assume $V = \{1, \dots, n\}$ and we use increasing chains of vertices $v_1 < \dots < v_n$ to represent their set.

Our construction will make use of some gadgets. Two gadgets can be composed to form a new one, this is done by “gluing” together two link points, namely, a sink and point is connected to a source link point forming transitions as depicted below

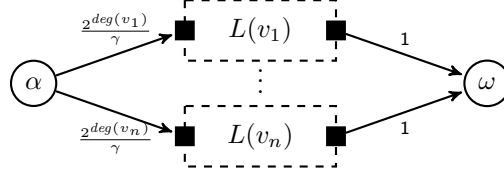


Moreover, a gadget can be rescaled by a factor $\varepsilon \in [0, 1]$, denoted as $M[\varepsilon]$ as



where β is a label that doesn't occur in M .

Now we are ready to introduce the construction of the SMMs that will be used in the proof. We start by describing the SMM $\mathcal{M}_{\mathcal{G}}$. In $\mathcal{M}_{\mathcal{G}}$ each state has residence time distribution κ and is labelled over the alphabet $V \cup \{\alpha, \omega\}$. Roughly, the purpose of $\mathcal{M}_{\mathcal{G}}$ is to allow the identification of each clique in \mathcal{G} by measuring certain word cylinders in $\mathcal{M}_{\mathcal{G}}$. The model $\mathcal{M}_{\mathcal{G}}$ consists of a start-state and an (absorbing) end-state, respectively labelled with α and ω , and n different gadgets $L(v)$ associated with each vertex $v \in V$. Each gadget $L(v)$ is connected with the start-state with entering probability $2^{\deg(v)}/\gamma$, where $\gamma = \sum_{v \in V} 2^{\deg(v)}$, and to the end-state with exit probability 1, as depicted below



The purpose of a gadget $L(v)$ is to measure with uniform probability all the cylinders of the form $\mathfrak{C}(s_1, \mathbb{R}_+, \dots, \mathbb{R}_+, s_k)$ for some $1 \leq k \leq n$, such that

- $\ell(s_1) < \dots < \ell(s_k)$ is an increasing sequence of vertices;
- for all $1 \leq i \leq k$, either $\ell(s_i) = v$ or $(v, \ell(s_i)) \in E$;
- $\ell(s_i) = v$ for some $1 \leq i \leq k$.

$L(v)$ is the sequential composition of gadgets $H_v(1) \dots H_v(n)$ where

$$H_v(u) = \begin{cases} \text{[Start] } \xrightarrow{1} (v) \xrightarrow{1} \text{[End]} & \text{if } u = v \\ \text{[Start] } \xrightarrow{\frac{1}{2}} (u) \xrightarrow{1} \text{[End]} & \text{if } (v, u) \in E \\ \text{[Start] } \xrightarrow{1} \text{[End]} & \text{otherwise} \end{cases}$$

Intuitively, the L -gadgets are used to count the number of vertices in a maximum clique of \mathcal{G} . Indeed, for any $v \in V$, it holds that

- all the paths in $L(v)$ have the same probability and,
- for each increasing sequence V' of vertices there is at most one path in $L(v)$ that generates it. In particular, those which are recognized always have v and cannot contain any vertex which is not adjacent to v in \mathcal{G} .

Note that the number of paths in any gadget $L(v)$ is $2^{\deg(v)}$. Therefore, $\mathcal{M}_{\mathcal{G}}$ measures with uniform probability, $1/\gamma$, each path from the start-state, say s , to the end-state. Therefore

$$\mathbb{P}_s^{\mathcal{M}_{\mathcal{G}}}(\alpha v_1 \dots v_k \omega) = k/\gamma \iff \{v_1, \dots, v_k\} \text{ is a clique in } \mathcal{G} \quad (7)$$

where, by abuse of notation, we denote with the string $\sigma_1 \cdots \sigma_n$ the word cylinder set $\mathfrak{C}(\ell^{-1}(\sigma_1), \mathbb{R}_+, \dots, \mathbb{R}_+, \ell^{-1}(\sigma_n))$.

Let now describe the construction of another SMM, denoted by \mathcal{M}_V . As before, each state in \mathcal{M}_V has residence time distribution κ and is labelled over the alphabet $V \cup \{\alpha, \omega\}$. The purpose of \mathcal{M}_V is to generate with uniform probability, $1/2^n$, all the increasing sequences of vertices in V . This is achieved by constructing \mathcal{M}_V as the following sequential composition of gadgets



Now we are ready to show that if one can compute δ in polynomial time, then he can solve Max Clique in polynomial time too. Let x_i be the number of word cylinders $w \in \alpha V^* \omega$ such that $\mathbb{P}_s^{\mathcal{M}_G}(w) = i/\gamma$. If we know the maximum j such that $x_j \neq 0$, by (7) we have that the maximum clique size of \mathcal{G} is j . Consider the vector \mathbf{x} , we show that, it represents the solution of a linear system of equations in n unknown where the i -th equation is constructed by cases on $1 \leq i \leq n$ as:

if $i2^n \leq \gamma$ we construct \mathcal{M}_i as the disjoint union of \mathcal{M}_G and $\mathcal{M}_V[i2^n/\gamma]$ ⁹ where s and s' are their respective start-points. Then the following hold

$$\begin{aligned} 2 \cdot \delta^{\mathcal{M}_i}(s, s') &= \sum_{E \in \mathcal{W}_{\mathcal{M}}} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)| \\ &= \mathbb{P}_{s'}^{\mathcal{M}}(\alpha\beta) + \sum_{w \in \alpha V^* \omega} |\mathbb{P}_s^{\mathcal{M}_G}(w) - \mathbb{P}_{s'}^{\mathcal{M}}(w)| \\ &= \left(1 - \frac{i2^n}{\gamma}\right) + \sum_{w \in \alpha V^* \omega} \left| \mathbb{P}_s^{\mathcal{M}_G}(w) - \frac{i2^n}{\gamma} \frac{1}{2^n} \right| \\ &= \left(1 - \frac{i2^n}{\gamma}\right) + \frac{1}{\gamma} \sum_{j=1}^n x_j |j - i| \end{aligned}$$

if $i2^n > \gamma$ we construct an SMM \mathcal{M}_i as the disjoint union of $\mathcal{M}_G[\gamma/i2^n]$ and \mathcal{M}_V where s and s' are their respective start-points. Then the following hold

$$\begin{aligned} 2 \cdot \delta^{\mathcal{M}_i}(s, s') &= \sum_{E \in \mathcal{W}_{\mathcal{M}}} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)| \\ &= \mathbb{P}_s^{\mathcal{M}}(\alpha\beta) + \sum_{w \in \alpha V^* \omega} |\mathbb{P}_s^{\mathcal{M}}(w) - \mathbb{P}_{s'}^{\mathcal{M}_V}(w)| \\ &= \left(1 - \frac{\gamma}{i2^n}\right) + \sum_{w \in \alpha V^* \omega} \left| \frac{\gamma}{i2^n} \mathbb{P}_s^{\mathcal{M}_G}(w) - \frac{1}{2^n} \right| \\ &= \left(1 - \frac{\gamma}{i2^n}\right) + \frac{1}{2^n} \sum_{j=1}^n x_j |j - i| \end{aligned}$$

⁹ When \mathcal{M} is not a gadget, its rescaling $\mathcal{M}[\varepsilon]$ is obtained by first swapping the start-state (resp. end-state) of \mathcal{M} with the source (resp. target) link point of the rescaling, then gluing as usual.

In summary, we obtain the linear system $\mathbf{b} = A\mathbf{x}$ with equations of the form

$$\begin{aligned} 2\gamma \cdot \delta^{\mathcal{M}_i}(s, s') + i2^n - \gamma &= \sum_{j=1}^n x_j |j - i| \quad \forall 1 \leq i \leq n. i2^n \leq \gamma \\ 2^{n+1} \cdot \delta^{\mathcal{M}_i}(s, s') + \frac{\gamma}{i} - 2^n &= \sum_{j=1}^n x_j |j - i| \quad \forall 1 \leq i \leq n. i2^n > \gamma \end{aligned} \quad (8)$$

One can notice that A is an invertible Toeplitz matrix. Therefore, provided that one has all the values $\delta^{\mathcal{M}_i}(s, s')$, we can efficiently solve $\mathbf{b} = A\mathbf{x}$ in $\Theta(n^2)$. Since the construction of each \mathcal{M}_i is in $O(\text{poly}(n))$, we conclude that computing δ for generic SMMs is **NP**-hard. \square

Proof (of Proposition 22). Let $\mathcal{G}(V, E)$ be an undirected graph with $|V| = n$. In Proof of Theorem 20 we showed that one can solve Max Clique by solving the Toeplitz system $\mathbf{b} = A\mathbf{x}$. Specifically, the size of the max clique of \mathcal{G} is $\|\mathbf{x}\|_\infty$.

Let define two column vectors $\mathbf{d}, \mathbf{c} \in \mathbb{R}^n$ and a diagonal matrix $D \in \mathbb{R}^{n \times n}$ as follows

$$d_i = \delta^{\mathcal{M}_i}(s, s') \quad c_i = \begin{cases} \frac{\gamma - i2^n}{2\gamma} & \text{if } i2^n \leq \gamma \\ \frac{1}{2} \left(1 - \frac{\gamma}{i2^{n+1}}\right) & \text{if } i2^n > \gamma \end{cases} \quad D_{i,i} = \begin{cases} \frac{1}{2\gamma} & \text{if } i2^n \leq \gamma \\ \frac{1}{2^{n+1}} & \text{if } i2^n > \gamma \end{cases}.$$

The system $\mathbf{b} = A\mathbf{x}$ (see (8) in Proof of Theorem 20) can be rewritten as $\mathbf{d} = D(A\mathbf{x}) + \mathbf{c}$.

Assume that there exists \mathbf{d}' that over-approximates \mathbf{d} within an absolute error $\epsilon > 0$, i.e., $\mathbf{d} \subseteq \mathbf{d}'$ and $\|\mathbf{d} - \mathbf{d}'\|_\infty \leq \epsilon$, and let \mathbf{x}' be the column vector that solves $\mathbf{d}' = D(A\mathbf{x}') + \mathbf{c}$. Then, the following equalities hold

$$\begin{aligned} \mathbf{d} - \mathbf{d}' &= D(A\mathbf{x}) + \mathbf{c} - D(A\mathbf{x}') + \mathbf{c} \\ &= DA(\mathbf{x} - \mathbf{x}'). \end{aligned}$$

Therefore $\mathbf{x} - \mathbf{x}' = A^{-1}D^{-1}(\mathbf{d} - \mathbf{d}')$ from which it follows that

$$\begin{aligned} \|\mathbf{x} - \mathbf{x}'\|_\infty &= \|A^{-1}D^{-1}(\mathbf{d} - \mathbf{d}')\|_\infty \\ &= \|A^{-1}\|_\infty \|D^{-1}\|_\infty \|\mathbf{d} - \mathbf{d}'\|_\infty \quad (\text{sub-multiplicativity}) \\ &\leq \|A^{-1}\|_\infty \|D^{-1}\|_\infty \epsilon. \quad (\text{by } \|\mathbf{d} - \mathbf{d}'\|_\infty \leq \epsilon) \end{aligned}$$

Since D is a diagonal matrix, its inverse is such that $D_{i,i}^{-1} = 1/D_{i,i}$, therefore $\|D^{-1}\|_\infty = n2^n$. Recalling that A is the Toeplitz matrix such that $A_{i,j} = |j - i|$, we let the reader verify that $\|A^{-1}\|_\infty = 1/(n-1)$. This implies that

$$|x_i - x'_i| \leq \frac{n2^n}{n-1} \epsilon \quad \forall 1 \leq i \leq n. \quad (9)$$

The inequality (9) states that assuming $\|\mathbf{d} - \mathbf{d}'\|_\infty \leq \epsilon$ we can obtain \mathbf{x}' (as the solution of the system $\mathbf{d}' = D(A\mathbf{x}') + \mathbf{c}$) that approximates the size of the max clique of \mathcal{G} within an absolute error $\frac{n2^n}{n-1}\epsilon$. Therefore, let x_j be the size of the max clique in \mathcal{G} , we can enforce x'_j to become an approximation of x_j within a multiplicative factor α , that is $x'_j \leq \alpha x_j$ by enforcing ϵ to hold $\epsilon \leq (\alpha - 1)x_j$.

Since $x_j \geq 1$, this requirement is implied by $\epsilon \leq (\alpha - 1)$. By this and (A) we obtain that

$$\epsilon \leq \frac{n-1}{n2^n}(1-\alpha). \quad (10)$$

Since the size of the SMMs used in the reduction is $\Theta(n^2)$ we have that an ϵ that satisfies (10) can be described by a function, say f , that depends on the size of the model and the given α . \square

Proof (of Lemma 30). First, note that $\eta: \Pi^2(S) \rightarrow \Pi(S) \times \Pi(S)$ is measurable (the proof is easy and we omit it). Let $C = (S', A', \tau', \rho', \ell')$ be a coupling for \mathcal{M} . We show that $\mathbb{P}_{s,s'}^C \# \eta \in \Omega(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}})$. To this end it suffices to prove that for every cylinder $C \in \mathfrak{C}(S, \Sigma_{\mathbb{R}_+})$, the following hold:

$$\mathbb{P}_{s,s'}^C \# \eta(C \times \Pi(S)) = \mathbb{P}_s^{\mathcal{M}}(C) \quad \text{and} \quad \mathbb{P}_{s,s'}^C \# \eta(\Pi(S) \times C) = \mathbb{P}_{s'}^{\mathcal{M}}(C).$$

We proceed by induction on the rank $n \geq 0$ of the cylinders $C \in \mathfrak{C}(S, \Sigma_{\mathbb{R}_+})$.

(Base case $n = 0$) Let $C = \mathfrak{C}(s_0)$, then we have that

$$\begin{aligned} \mathbb{P}_{s,s'}^C \# \eta(\mathfrak{C}(s_0) \times \Pi(S)) &= \mathbb{P}_{s,s'}^C(\mathfrak{C}(\{s_0\} \times S)) \\ &= \chi_{\{(s,s')\}}(\{s_0\} \times S) \\ &= \chi_{\{s\}}(s_0) = \mathbb{P}_s^{\mathcal{M}}(\mathfrak{C}(s_0)). \end{aligned}$$

(Inductive step $n \geq 0$) Let $C = \mathfrak{C}(s_0, R_0, \dots, R_n, s_{n+1})$, then we have

$$\begin{aligned} \mathbb{P}_{s,s'}^C \# \eta(\mathfrak{C}(s_0, R_0, \dots, R_n, s_{n+1}) \times \Pi(S)) &= \\ &= \mathbb{P}_{s,s'}^C(\mathfrak{C}(\{s_0\} \times S, R_0 \times \mathbb{R}_+, \dots, R_n \times \mathbb{R}_+, \{s_{n+1}\} \times S)) \\ &= \mathbb{P}_{s,s'}^C \# \eta(\mathfrak{C}(s_0, \dots, s_n) \times \Pi(S)) \cdot \sum_{u \notin A} P'((s_n, u), R_n \times \mathbb{R}_+, \{s_{n+1}\} \times S) \\ &= \mathbb{P}_s^{\mathcal{M}}(\mathfrak{C}(s_0, \dots, s_n)) \cdot \sum_{u \notin A} P'((s_n, u), R_n \times \mathbb{R}_+, \{s_{n+1}\} \times S). \end{aligned}$$

If $s_n \in A$, the summation $\sum_{u \notin A} P'((s_n, u), R_n \times \mathbb{R}_+, \{s_{n+1}\} \times S) = 0$, and we are done, otherwise

$$\begin{aligned} &= \mathbb{P}_s^{\mathcal{M}}(\mathfrak{C}(s_0, \dots, s_n)) \cdot \sum_{u \notin A} \rho'(s_n, u)(R_n \times \mathbb{R}_+) \cdot \tau'(s_n, u)(\{s_{n+1}\} \times S) \\ &= \mathbb{P}_s^{\mathcal{M}}(\mathfrak{C}(s_0, \dots, s_n)) \cdot \sum_{u \notin A} \rho(s_n)(R_n) \cdot \tau(s_n)(s_{n+1}) \\ &= \mathbb{P}_s^{\mathcal{M}}(\mathfrak{C}(s_0, R_0, \dots, R_{n-1}, s_n)). \end{aligned}$$

The right marginal follows similarly. \square

Proof (of Lemma 25). (\Rightarrow) Assume $\pi \equiv_{\ell}^{\omega} \pi'$ and let $C \in \mathcal{T}_{\mathcal{M}}$. We prove that $\pi \in C$ implies $\pi' \in C$. Let $C = \mathfrak{C}([s_0]_{\equiv_{\ell}}, R_0, \dots, R_{n-1}, [s_n]_{\equiv_{\ell}})$ for some $s_i \in S$ and $R_i \in \Sigma_{\mathbb{R}_+}$ ($i = 0..n$). By $\pi \in C$ we have that, for all $i \leq n$, $\pi[i] \equiv_{\ell} s_i$ and, for all $i < n$, $\pi[i] \in R_i$. From $\pi \equiv_{\ell}^{\omega} \pi'$, we have that $\pi[j] \equiv_{\ell} \pi'[j]$ and $\pi[j] = \pi'[j]$ for all $j \in \mathbb{N}$. Hence $\pi' \in C$.

(\Leftarrow) We proceed by contraposition, proving that if $\pi \equiv_{\ell}^{\omega} \pi'$ then there exists a cylinder $C \in \mathcal{T}_{\mathcal{M}}$ such that $\pi \in C$ and $\pi' \notin C$. By $\pi \equiv_{\ell}^{\omega} \pi'$ we have that $\pi[k] \not\equiv_{\ell} \pi'[k]$ or $\pi[k] \neq \pi'[k]$ for some $k \in \mathbb{N}$.

If $\pi[k] \not\equiv_\ell \pi'[k]$ holds we pick $C := \mathfrak{C}([\pi[0]]_{\equiv_\ell}, \{\pi\langle 0 \rangle\}, \dots, \{\pi\langle k-1 \rangle\}, [\pi[k]]_{\equiv_\ell})$; if $\pi\langle k \rangle \neq \pi'\langle k \rangle$ holds we pick $C := \mathfrak{C}([\pi[0]]_{\equiv_\ell}, \{\pi\langle 0 \rangle\}, \dots, \{\pi\langle k \rangle\}, [\pi[k+1]]_{\equiv_\ell})$. In both cases $C \in \mathcal{T}_\mathcal{M}$, $\pi \in C$, and $\pi' \notin C$. \square

Proposition 43. *Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$ be an SMM, $E \subseteq \Pi(S)$, and define $E@k \subseteq \Pi(S)$ by induction on $k \in \mathbb{N}$ as follows*

$$E@0 = E, \quad E@(k+1) = tl_{\mathcal{M}}^{-1}(E@k).$$

Then, for all $k \in \mathbb{N}$, $E@k = \{\pi \mid \pi|_k \in E\}$, and if $E \in \Sigma_\mathcal{M}$, then $E@k \in \Sigma_\mathcal{M}$.

Proof. We proceed by induction on $k \geq 0$.

(Base case: $k = 0$) Trivial.

(Inductive step: $k \geq 0$) $E@(k+1) = tl_{\mathcal{M}}^{-1}(E@k)$. By inductive hypothesis $E@k \in \Sigma_\mathcal{M}$, so that, by Lemma 39, $tl_{\mathcal{M}}^{-1}(E@k) \in \Sigma_\mathcal{M}$. Moreover,

$$\begin{aligned} E@(k+1) &= tl_{\mathcal{M}}^{-1}(E@k) && (\text{def. @}) \\ &= tl_{\mathcal{M}}^{-1}(\{\pi \in \Pi(S) \mid \pi|_k \in E\}) && (\text{ind. hp.}) \\ &= \{\pi \in \Pi(S) \mid \pi|_{k+1} \in E\} && (\text{def. } tl_{\mathcal{M}}) \end{aligned}$$

\square

Proof (of Lemma 26). We characterize $\not\equiv_\ell^\omega \subseteq \Pi(S) \times \Pi(S)$ as a countable union of measurable rectangles in $\Sigma_\mathcal{M} \otimes \Sigma_\mathcal{M}$.

Let $DiffS(k)$ and $DiffT(k)$ be defined, for $k \geq 0$, as follows

$$\begin{aligned} DiffS(k) &:= \bigcup_{C \in S/\equiv_\ell} \mathfrak{C}(C)@k \times \mathfrak{C}(S \setminus C)@k, \\ DiffT(k) &:= \bigcup_{t, t' \in \mathbb{Q}_+} \mathfrak{C}(S, (t, t'), S)@k \times \mathfrak{C}(S, \mathbb{R}_+ \setminus (t, t'), S)@k. \end{aligned}$$

where, for $E \subseteq \Pi(S)$, $E@k$ is defined as in Proposition 43. For any equivalence class $C \in S/\equiv_\ell$, the cylinders $\mathfrak{C}(C)$ and $\mathfrak{C}(S \setminus C)$ are in $\Sigma_\mathcal{M}$; the same holds for $\mathfrak{C}(S, (t, t'), S)$ and $\mathfrak{C}(S, \mathbb{R}_+ \setminus (t, t'), S)$, for all $t, t' \in \mathbb{Q}_+$. Therefore, by Proposition 43, for all $k \in \mathbb{N}$, $DiffS(k)$ and $DiffT(k)$ are a countable union of rectangles in $\Sigma_\mathcal{M} \otimes \Sigma_\mathcal{M}$. Now we show that $\not\equiv_\ell^\omega = \bigcup_{k \in \mathbb{N}} (DiffS(k) \cup DiffT(k))$.

(\subseteq) Let $\pi \not\equiv_\ell^\omega \pi'$. By definition $\ell^\omega(\pi) \neq \ell^\omega(\pi')$, that is, there exists an index $k \in \mathbb{N}$ such that $\pi[k] \not\equiv_\ell \pi'[k]$ or $\pi\langle k \rangle \neq \pi'\langle k \rangle$. If $\pi[k] \not\equiv_\ell \pi'[k]$, by Proposition 43, $(\pi, \pi') \in \mathfrak{C}([\pi[k]]_{\equiv_\ell})@k \times \mathfrak{C}(S \setminus [\pi[k]]_{\equiv_\ell})@k$.

If $\pi\langle k \rangle \neq \pi'\langle k \rangle$ holds, let $\varepsilon = |\pi\langle k \rangle - \pi'\langle k \rangle|$. Since \mathbb{Q}_+ is dense in \mathbb{R}_+ , every open set has nonempty intersection with \mathbb{Q}_+ , so that there exist $t \in \mathbb{Q}_+ \cap (\pi\langle k \rangle - \varepsilon, \pi\langle k \rangle)$ and $t' \in \mathbb{Q}_+ \cap (\pi\langle k \rangle, \pi\langle k \rangle + \varepsilon)$. Clearly, $\pi\langle k \rangle \in (t, t')$ and $\pi'\langle k \rangle \notin (t, t')$, therefore, by Proposition 43, $(\pi, \pi') \in \mathfrak{C}(S, (t, t'), S)@k \times \mathfrak{C}(S, \mathbb{R}_+ \setminus (t, t'), S)@k$.

(\supseteq) Let $(\pi, \pi') \in \bigcup_{k \in \mathbb{N}} (DiffS(k) \cup DiffT(k))$, by construction there exists $k \in \mathbb{N}$ such that $(\pi, \pi') \in DiffS(k)$ or $(\pi, \pi') \in DiffT(k)$.

$$\begin{aligned} DiffS(k) &= \bigcup_{C \in S/\equiv_\ell} \mathfrak{C}(C)@k \times \mathfrak{C}(S \setminus C)@k && (\text{by def.}) \\ &= \bigcup_{C \in S/\equiv_\ell} \{(\pi, \pi') \mid \pi \in \mathfrak{C}(C)@k \text{ and } \pi' \in \mathfrak{C}(S \setminus C)@k\} \\ &= \{(\pi, \pi') \mid \pi[k] \not\equiv_\ell \pi'[k]\}, && (\text{by Prop. 43}) \end{aligned}$$

$$\begin{aligned}
DiffT(k) &= \bigcup_{t,t' \in \mathbb{Q}_+} \mathfrak{C}(S, (t, t'), S) @k \times \mathfrak{C}(S, \mathbb{R}_+ \setminus (t, t'), S) @k \\
&= \bigcup_{t,t' \in \mathbb{Q}_+} \left\{ (\pi, \pi') \mid \begin{array}{l} \pi \in \mathfrak{C}(S, (t, t'), S) @k \text{ and} \\ \pi' \in \mathfrak{C}(S, \mathbb{R}_+ \setminus (t, t'), S) @k \end{array} \right\} \\
&= \bigcup_{t,t' \in \mathbb{Q}_+} \{(\pi, \pi') \mid \pi\langle k \rangle \in (t, t') \text{ and } \pi'\langle k \rangle \notin (t, t')\} \quad (\text{by Prop. 43}) \\
&\subseteq \{(\pi, \pi') \mid \pi\langle k \rangle \neq \pi'\langle k \rangle\}.
\end{aligned}$$

Therefore, there exists an index $k \in \mathbb{N}$ such that $\pi[k] \not\equiv_{\ell} \pi'[k]$ or $\pi\langle k \rangle \neq \pi'\langle k \rangle$. Thus, $\ell^{\omega}(\pi) \neq \ell^{\omega}(\pi')$, that is $\pi \not\equiv_{\ell^{\omega}} \pi'$. \square

Proof (of Lemma 27). We first prove that $\sup_{E \in \Sigma_{\mathcal{M}}} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)|$ is a lower bound for $\{\omega(\not\equiv_{\ell^{\omega}}) \mid \omega \in \Omega(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}})\}$. Let $\omega \in \Omega(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}})$ and $E \in \Sigma_{\mathcal{M}}$, then

$$\begin{aligned}
\mathbb{P}_s^{\mathcal{M}}(E) &= \omega(E \times \Pi(S)) && (\text{by } \omega \in \Omega(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}})) \\
&\geq \omega(\Pi(S) \times E \cap \equiv_{\ell^{\omega}}) && (\text{by Lemma 25}) \\
&= 1 - \omega((\Pi(S) \times E)^c \cup \not\equiv_{\ell^{\omega}}) && (\text{by complement}) \\
&\geq 1 - \omega((\Pi(S) \times E)^c) - \omega(\not\equiv_{\ell^{\omega}}) && (\text{sub additivity}) \\
&= \omega(\Pi(S) \times E) - \omega(\not\equiv_{\ell^{\omega}}) && (\text{by complement}) \\
&= \mathbb{P}_{s'}^{\mathcal{M}}(E) - \omega(\not\equiv_{\ell^{\omega}}). && (\text{by } \omega \in \Omega(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}}))
\end{aligned}$$

Thus, by the generality of $\omega \in \Omega(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}})$ and $E \in \Sigma_{\mathcal{M}}$, it immediately follows that $\sup_{E \in \Sigma_{\mathcal{M}}} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)| \leq \min \{\omega(\not\equiv_{\ell^{\omega}}) \mid \omega \in \Omega(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}})\}$.

Now we prove that there exists an optimal coupling $\omega^* \in \Omega(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}})$ such that $\omega^*(\not\equiv_{\ell^{\omega}}) = \sup_{E \in \Sigma_{\mathcal{M}}} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)|$. The following proof is a slightly modification of [23, Theorem 5.2]. If $\mathbb{P}_s^{\mathcal{M}} = \mathbb{P}_{s'}^{\mathcal{M}}$, ω^* is the measure that assigns 0 to all measurable sets contained in $\equiv_{\ell^{\omega}}$ and 1 otherwise. Let $\mathbb{P}_s^{\mathcal{M}} \neq \mathbb{P}_{s'}^{\mathcal{M}}$. For each $\equiv_{\ell^{\omega}}$ -equivalence class $C \in \Pi(S)/\equiv_{\ell^{\omega}}$, fix a representative element $C^* \in C$, and define $\psi: \Pi(S) \rightarrow \Pi(S) \times \Pi(S)$ by $\psi(\pi) = ([\pi]_{\equiv_{\ell^{\omega}}}^*, [\pi]_{\equiv_{\ell^{\omega}}})$. We prove that ψ is measurable from $(\Pi(S), \Sigma_{\mathcal{M}})$ to $(\Pi(S), \Sigma_{\mathcal{M}}) \otimes (\Pi(S), \Sigma_{\mathcal{M}})$. To do this it suffices to show that for all $E, E' \in \Sigma_{\mathcal{M}}$, $\psi^{-1}(E \times E') \in \Sigma_{\mathcal{M}}$. This follows by Lemma 25:

$$\begin{aligned}
\psi^{-1}(E \times E') &= \{\pi \mid ([\pi]_{\equiv_{\ell^{\omega}}}^*, [\pi]_{\equiv_{\ell^{\omega}}}) \in E \times E'\} && (\text{by def. } \psi) \\
&= \{\pi \mid [\pi]_{\equiv_{\ell^{\omega}}}^* \in E \text{ and } [\pi]_{\equiv_{\ell^{\omega}}} \in E'\} \\
&= \{\pi \mid \pi \in E \text{ and } \pi \in E'\} && (\text{by Lemma 25}) \\
&= E \cap E' \in \Sigma_{\mathcal{M}}.
\end{aligned}$$

Moreover $\psi^{-1}(\equiv_{\ell^{\omega}}) = \Pi(S)$, since all timed paths have their $\equiv_{\ell^{\omega}}$ -representative.

Define the measure $\lambda: \Sigma_{\mathcal{M}} \rightarrow \mathbb{R}_+$ as $\lambda = \mathbb{P}_s^{\mathcal{M}} + \mathbb{P}_{s'}^{\mathcal{M}}$, and the functions $g, g': \Pi(S) \rightarrow \mathbb{R}_+$ as the Radon-Nikodym derivatives of $\mathbb{P}_s^{\mathcal{M}}$ and $\mathbb{P}_{s'}^{\mathcal{M}}$, respectively, w.r.t. λ , that is, $g = d\mathbb{P}_s^{\mathcal{M}}/d\lambda$ and $g' = d\mathbb{P}_{s'}^{\mathcal{M}}/d\lambda$. Note that g and g' are well defined, since $\mathbb{P}_s^{\mathcal{M}}$ and $\mathbb{P}_{s'}^{\mathcal{M}}$ are both absolutely continuous w.r.t. λ . Now, denote by $g \wedge g'$ the point-wise meet of g and g' (i.e., $(g \wedge g')(\pi) = \min \{g(\pi), g'(\pi)\}$), and define the measures $\mu: \Sigma_{\mathcal{M}} \rightarrow [0, 1]$ and $\mu^*: \Sigma_{\mathcal{M}} \otimes \Sigma_{\mathcal{M}}$ as follows

$$d\mu/d\lambda = g \wedge g', \quad \mu^* = \mu \# \psi.$$

Since $\psi^{-1}(\equiv_{\ell}^{\omega}) = \Pi(S)$, μ^* puts all its mass in \equiv_{ℓ}^{ω} . Call this mass $\gamma = \mu^*(\equiv_{\ell}^{\omega})$, and define $\nu, \nu': \Sigma_{\mathcal{M}} \rightarrow \mathbb{R}$ and $\omega^*: \Sigma_{\mathcal{M}} \otimes \Sigma_{\mathcal{M}} \rightarrow [0, 1]$ as follows

$$\nu = \mathbb{P}_s^{\mathcal{M}} - \mu, \quad \nu' = \mathbb{P}_{s'}^{\mathcal{M}} - \mu, \quad \omega^* = \frac{\nu \times \nu'}{1 - \gamma} + \mu^*$$

Note that, by the assumption that $\mathbb{P}_s^{\mathcal{M}} \neq \mathbb{P}_{s'}^{\mathcal{M}}$, we have $\gamma < 1$, so that ω^* is well defined and, in particular, $\omega^*(\equiv_{\ell}^{\omega}) = \gamma$. Now we show that $\omega^* \in \Omega(\mathbb{P}_s^{\mathcal{M}}, \mathbb{P}_{s'}^{\mathcal{M}})$. Let $E \in \Sigma_{\mathcal{M}}$, then

$$\begin{aligned} \omega^*(E \times \Pi(S)) &= \frac{\nu(E) \cdot \nu'(\Pi(S))}{1 - \gamma} + \mu^*(E \times \Pi(S)) && \text{(by def. } \omega^*) \\ &= \frac{\nu(E) \cdot (\mathbb{P}_{s'}^{\mathcal{M}}(\Pi(S)) - \mu(\Pi(S)))}{1 - \gamma} + \mu^*(E \times \Pi(S)) && \text{(by def. } \nu) \\ &= \frac{\nu(E) \cdot (1 - \gamma)}{1 - \gamma} + \mu^*(E \times \Pi(S)) && \text{(by def. } \mu^*) \\ &= \mathbb{P}_s^{\mathcal{M}}(E) - \mu(E) + \mu^*(E \times \Pi(S)) && \text{(by def. } \nu) \\ &= \mathbb{P}_s^{\mathcal{M}}(E) - \mu(E) + \mu(E) && \text{(by def. } \mu^*) \\ &= \mathbb{P}_s^{\mathcal{M}}(E). \end{aligned}$$

Similarly $\omega^*(\Pi(S) \times E) = \mathbb{P}_{s'}^{\mathcal{M}}(E)$, so that the marginals are correct and we indeed have a proper coupling. The following shows that ω^* is optimal

$$\begin{aligned} \sup_{E \in \Sigma_{\mathcal{M}}} |\mathbb{P}_s^{\mathcal{M}}(E) - \mathbb{P}_{s'}^{\mathcal{M}}(E)| &= \sup_{E \in \Sigma_{\mathcal{M}}} \left| \int_E g \, d\lambda - \int_E g' \, d\lambda \right| && \text{(Radon-Nikodym)} \\ &= \sup_{E \in \Sigma_{\mathcal{M}}} \int_E |g - g'| \, d\lambda && \text{(linearity)} \\ &= \frac{1}{2} \int_{\Pi(S)} |g - g'| \, d\lambda && \text{(Jordan-Hahn decomposition)} \\ &= \frac{1}{2} \cdot 2 \left(1 - \int_{\Pi(S)} g \wedge g' \, d\lambda \right) && (*) \\ &= 1 - \mu(\Pi(S)) && \text{(Radon-Nikodym)} \\ &= 1 - \gamma && \text{(def. } \gamma) \\ &= 1 - \omega^*(\equiv_{\ell}^{\omega}) && \text{(def. } \omega^* \text{ and } (\nu \times \nu')(\equiv_{\ell}^{\omega}) = 0) \\ &= \omega^*(\not\equiv_{\ell}^{\omega}) && \text{(complement)} \end{aligned}$$

where (*) follows since $2 - \int |g - g'| \, d\lambda = 2 \int g \wedge g' \, d\lambda$ (this can be understood considering the geometrical interpretation of integral as “the area below a function”). This ends the proof, showing that the minimum element exists. \square

Proof (of Lemma 32). Let the variable $p_{s,s'}$ denote $(\mathbb{P}_{s,s'}^{\mathcal{C}} \# \eta)(\not\equiv_{\ell}^{\omega})$. Our goal is to compute $p_{s,s'}$ for all $(s, s') \in S'$. Lemma 26 characterizes $\not\equiv_{\ell}^{\omega}$ as the set of all pairs of times paths (π, π') s.t. $\pi[k] \not\equiv_{\ell} \pi'[k]$ or $\pi\langle k \rangle \neq \pi'\langle k \rangle$ for some $k \in \mathbb{N}$.

Let consider the set of states $B = \{(u, v) \in S' \mid u \neq v\}$. Clearly, if B is not reachable from (s, s') in the underlying graph of \mathcal{C} , then $p_{s,s'} = 0$. Note that this is the case when $s \equiv s'$ and $(s, s') \in A'$, that is to say $s \equiv s$ and $s, s' \in A$.

Assume $s \neq s'$, clearly we have that $p_{s,s'} = 1$ since $(s, s') \in B$.

Otherwise, if $(s, s') \notin B \cup A'$, that is to say $s \equiv s'$ and $s, s' \notin A$, we have that the probability that a coupled is performed from (s, s') at different time points is the probability associated by ρ' to the event $\{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x \neq y\}$, denoted by $\rho'(\neq)$. Therefore we have that the following equation holds

$$p_{s,s'} = \rho'(\neq) + (1 - \rho'(\neq)) \cdot \sum_{(u,v) \in S'} \tau'(s, s')(u, v) \cdot p_{u,v}.$$

The equation states that $p_{s,s'}$ is the probability that one of the two following mutually exclusive events occurs:

- (i) a coupled step from (s, s') occurs at different coupled time points; or
- (ii) the same occurs starting from some other state $(u, v) \in S'$, after moving there from (s, s') at the same coupled time points. \square

Proof (of Lemma 34). Assume $\mathcal{M} = (S, A, \tau, \rho, \ell)$ and $\mathcal{C} = (S', A', \tau', \rho', \ell')$. To prove $f^{\mathcal{M}} \sqsubseteq d$, it suffices to show that $F^{\mathcal{M}}(d) \sqsubseteq d$. Indeed, by Tarski's fixed point theorem, $f^{\mathcal{M}}$ is a lower bound of $\{d \mid F^{\mathcal{M}}(d) \sqsubseteq d\}$. Let $s, s' \in S$, then

1. if $s \neq s'$, then $F^{\mathcal{M}}(d)(s, s') = 1 = I^{\mathcal{C}}(d)(s, s') = d(s, s')$;
2. if $s \equiv s'$ and $s, s' \in A$ then $F^{\mathcal{M}}(d)(s, s') = 0 = I^{\mathcal{C}}(d)(s, s') = d(s, s')$;
3. otherwise, note that for any $0 \leq \alpha \leq \alpha' \leq 1$ and $0 \leq \beta \leq \beta' \leq 1$ the following hold

$$\begin{aligned} \alpha + (1 - \alpha)\beta &\leq \alpha + (1 - \alpha)\beta' && (\beta \leq \beta') \\ &= \beta' - \beta' + \alpha + (1 - \alpha)\beta' \\ &= \beta' - \alpha\beta' - (1 - \alpha)\beta' + \alpha + (1 - \alpha)\beta' && (0 \leq \alpha \leq 1) \\ &= \beta' - \alpha\beta' + \alpha = \beta' + (1 - \beta')\alpha \\ &\leq \beta' + (1 - \beta')\alpha' && (\alpha \leq \alpha') \\ &= \alpha' + (1 - \alpha')\beta'. && (0 \leq \beta \leq 1) \end{aligned}$$

Moreover, the two following inequalities hold:

$$\begin{aligned} \|\rho(s) - \rho(s')\|_{\text{TV}} &= \inf \{ \bar{\rho}(\neq_{\mathbb{R}_+}) \mid \bar{\rho} \in \Omega(\rho(s), \rho(s')) \} \quad (\text{by [23, Th. 5.2]}) \\ &\leq \rho'(s, s')(\neq_{\mathbb{R}_+}), \quad (\rho'(s, s') \in \Omega(\rho(s), \rho'(s'))) \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}_d(\tau(s), \tau(s')) &= \min_{\omega \in \Omega(\tau(s), \tau(s'))} \sum_{u,v \in S} d(u, v) \cdot \omega(u, v) \quad (\text{by def.}) \\ &\leq \sum_{u,v \in S} d(u, v) \cdot \tau'(s, s')(u, v) \cdot (\tau'(s, s') \in \Omega(\tau(s), \tau(s'))) \end{aligned}$$

From the above we have

$$\begin{aligned}
F^{\mathcal{M}}(d)(s, s') &= \\
&= \|\rho(s) - \rho(s')\|_{\text{TV}} + (1 - \|\rho(s) - \rho(s')\|_{\text{TV}}) \mathcal{K}_d(\tau(s), \tau(s')) \quad (\text{by def. } F^{\mathcal{M}}) \\
&\leq \rho'(s, s')(\neq_{\mathbb{R}_+}) + (1 - \rho'(s, s')(\neq_{\mathbb{R}_+})) \sum_{u, v \in S} d(u, v) \cdot \tau'(s, s')(u, v) \\
&= \Gamma^{\mathcal{C}}(d)(s, s') \quad (\text{by def. } \Gamma^{\mathcal{C}}) \\
&= d(s, s'). \quad (\text{by hypothesis})
\end{aligned}$$

This proves that $F^{\mathcal{M}}(d) \subseteq d$. \square

Proof (of Lemma 35). We firstly show that, for any fixed $d: S \times S \rightarrow [0, 1]$, there exists a coupling $\mathcal{C} = (S', A', \tau', \rho', \ell')$ for \mathcal{M} such that $\Gamma^{\mathcal{C}}(d) = F^{\mathcal{M}}(d)$. We construct \mathcal{C} as follows: for each $s, s' \in S$ such that $s \equiv_{\ell} s'$, $s, s' \notin A$, applying [23, Theorem 5.2], we can fix $\rho'(s, s')$ as the coupling in $\Omega(\rho(s), \rho(s'))$ such that $\rho'(s, s')(\neq_{\mathbb{R}_+}) = \|\rho(s) - \rho(s')\|_{\text{TV}}$; and we fix $\tau'(s, s')$ as one vertex of the transportation polytope $\Omega(\tau(s), \tau(s'))$ that achieves the value $\mathcal{K}_d(\tau(s), \tau(s'))$. The sets S' , A' , and the function ℓ' are fixed according to Definition 28. For such a coupling \mathcal{C} , its easy to verify that $\Gamma^{\mathcal{C}}(d) = F^{\mathcal{M}}(d)$ holds.

Let \mathcal{D} be a coupling for \mathcal{M} such that $\Gamma^{\mathcal{D}}(f^{\mathcal{M}}) = F^{\mathcal{M}}(f^{\mathcal{M}})$. By definition $F^{\mathcal{M}}(f^{\mathcal{M}}) = f^{\mathcal{M}}$, therefore $f^{\mathcal{M}}$ is a fixed point for $\Gamma^{\mathcal{D}}$. By Lemma 34, $f^{\mathcal{M}}$ is a lower bound of the set of fixed points of $\Gamma^{\mathcal{D}}$, so that $f^{\mathcal{M}} = \gamma^{\mathcal{D}}$. By Lemma 34, we have also that, for any coupling \mathcal{C} for \mathcal{M} , $f^{\mathcal{M}} \subseteq \gamma^{\mathcal{C}}$. Therefore, for $D = \{\gamma^{\mathcal{C}} \mid \mathcal{C} \text{ coupling for } \mathcal{M}\}$, we have that $f^{\mathcal{M}} \in D$ and $f^{\mathcal{M}}$ is a lower bound for D . Hence, $f^{\mathcal{M}} = \min D$. Then, the thesis follows by Corollary 33. \square

Proof (of Theorem 36). By Lemma 35 it suffices to show the two points for $f^{\mathcal{M}}$.

1. By showing that $F^{\mathcal{M}}$ is ω -continuous and proving that, if $d: S \times S \rightarrow [0, 1]$ is a pseudometric then is so $F^{\mathcal{M}}(d)$.
2. (\Rightarrow) We prove that $R = \{(s, s') \mid f^{\mathcal{M}}(s, s') = 0\}$ is a bisimulation on \mathcal{M} . Clearly, R is an equivalence. Assume $(s, s') \in R$. By definition of $F^{\mathcal{M}}$, it holds:

- (a) $s \equiv s'$ and $s, s' \in A$, or
- (b) $s \equiv s'$ and $s, s' \notin A$, and $\alpha + (1 - \alpha)\mathcal{K}_{f^{\mathcal{M}}}(\tau(s), \tau(s')) = 0$ where $\alpha = \|\rho(s) - \rho(s')\|_{\text{TV}}$.

If (2b) holds, we have $\|\rho(s) - \rho(s')\|_{\text{TV}} = 0$ and $\mathcal{K}_{f^{\mathcal{M}}}(\tau(s), \tau(s')) = 0$. The total variation distance is a metric in $\Delta(\mathbb{R}_+)$, therefore $\rho(s) = \rho(s')$. By [17, Lemma 3.1], $\mathcal{K}_{f^{\mathcal{M}}}(\tau(s), \tau(s')) = 0$ implies that, for all $C \in S/R$, $\tau(s)(C) = \tau(s')(C)$. Therefore R is a bisimulation.

(\Leftarrow) Let R be a bisimulation on \mathcal{M} , and define $d_R: S \times S \rightarrow [0, 1]$ by $d_R(s, s') = 0$ if $(s, s') \in R$ and $d_R(s, s') = 1$ if $(s, s') \notin R$, for all $s, s' \in S$. We prove that $F^{\mathcal{M}}(d_R) \subseteq d_R$. If $(s, s') \notin R$, then $d_R(s, s') = 1 \geq F^{\mathcal{M}}(d_R)(s, s')$. If $(s, s') \in R$, then $\ell(s) = \ell(s')$ and one of the following holds:

- (a) $s, s' \in A$, or
- (b) $s, s' \notin A$, $\rho(s) = \rho(s')$ and, $\forall C \in S/R$, $\tau(s)(C) = \tau(s')(C)$.

If (2a) holds, then $F^{\mathcal{M}}(d_R)(s, s') = 0 = d_R(s, s')$.
 If (2b) holds, $\|\rho(s) - \rho(s')\|_{\text{TV}} = 0$ and by [17, Lemma 3.1] and the fact that, for all $C \in S/R$, $\tau(s)(C) = \tau(s')(C)$, we have $\mathcal{K}_d(\tau(s), \tau(s')) = 0$. Therefore, $F^{\mathcal{M}}(d_R)(s, s') = 0 = d_R(s, s')$. By the generality of the chosen R and Tarski's fixed point theorem, we have that $s \sim_{\mathcal{M}} s'$ implies $f^{\mathcal{M}}(s, s') = 0$. \square

Lemma 44. *Let \mathcal{M} be an SMM and let $G^{\mathcal{M}}: [0, 1]^{S \times S} \rightarrow [0, 1]^{S \times S}$ be defined by $G(d)(s, s') = 0$ if $s \sim_{\mathcal{M}} s'$, and $G(d)(s, s') = F^{\mathcal{M}}(d)(s, s')$ otherwise. Then, G has a unique fixed point, and it corresponds to $f^{\mathcal{M}}$.*

Proof (of Lemma 44). Let $\mathcal{M} = (S, A, \tau, \rho, \ell)$. The proof follows the same idea of [10, Proposition 17 and Corollary 18].

We first prove that G has a unique fixed point. Since $F^{\mathcal{M}}$ is monotone, it can be easily deduced that G is monotone as well. By Tarski's fixed point theorem, G has a least and a greatest fixed point. Therefore it suffices to prove that if $d \sqsubseteq d'$ are both fixed point of G then $d = d'$. Let

$$m = \max_{s, s' \in S} \{d'(s, s') - d(s, s')\}, \quad s M s' \iff d'(s, s') - d(s, s') = m.$$

We show that $m = 0$, that is $d = d'$. Assume $s M s'$, we distinguish 3 cases:

1. if $s \sim_{\mathcal{M}} s'$, then $d'(s, s') - d(s, s') = 0 - 0 = 0$. Note this covers also the case when $s \equiv s'$ and $s, s' \in A$.
2. if $s \not\sim s'$, then $d'(s, s') - d(s, s') = F^{\mathcal{M}}(d')(s, s') - F^{\mathcal{M}}(d)(s, s') = 1 - 1 = 0$.
3. otherwise, we have $s \equiv s'$ but $s \not\sim_{\mathcal{M}} s'$ (note this implies $s, s' \notin A$). Let $\alpha = \|\rho(s) - \rho(s')\|_{\text{TV}}$ and assume that $G(d)(s, s')$ is achieved on $\omega \in \Omega(\tau(s), \tau(s'))$, i.e., $G(d)(s, s') = \alpha + (1 - \alpha) \sum_{u, v \in S} d(u, v) \cdot \omega(u, v)$. Then

$$\begin{aligned} m &= d'(s, s') - d(s, s') && \text{(by } s M s') \\ &= G(d')(s, s') - G(d)(s, s') && \text{(by hp. on } d \text{ and } d') \\ &= (1 - \alpha)(\mathcal{K}_{d'}(\tau(s), \tau(s')) - \sum_{u, v \in S} d(u, v) \cdot \omega(u, v)) && \text{(by hp. on } \omega) \\ &\leq (1 - \alpha) \left(\sum_{u, v \in S} d'(u, v) \cdot \omega(u, v) - \sum_{u, v \in S} d(u, v) \cdot \omega(u, v) \right) && \text{(by def. } \mathcal{K}) \\ &= (1 - \alpha) \left(\sum_{u, v \in S} (d'(u, v) - d(u, v)) \cdot \omega(u, v) \right) \end{aligned}$$

By hypothesis on m and ω we have respectively that $d'(s, s') - d(s, s') \leq m$, and $\sum_{u, v \in S} \omega(u, v) = 1$, thus, from the above inequality, that is

$$m \leq (1 - \alpha) \left(\sum_{u, v \in S} (d'(u, v) - d(u, v)) \cdot \omega(u, v) \right), \quad (11)$$

we have that

- if $0 < \alpha \leq 1$, then $0 \leq 1 - \alpha < 1$. Since the left hand side of (11) is bounded by $(1 - \alpha)m$ we have that (11) holds only for $m = 0$;

- if $\alpha = 0$ we have that (11) holds only if $d'(u, v) - d(u, v) = m$ whenever $\omega(u, v) > 0$. Thus ω has support contained in M . This implies that there exists a coupling model \mathcal{C} for \mathcal{M} (constructed using ω) such that $\gamma^{\mathcal{C}}(s, s') = 0$. By Lemma 35 and Theorem 36 we have that $s \sim_{\mathcal{M}} s'$ therefore $m = 0$.

It remains to prove that $f^{\mathcal{M}}$ is a fixed point for G , that is $f^{\mathcal{M}} = G(f^{\mathcal{M}})$. On the one hand, suppose that $s \sim_{\mathcal{M}} s'$. Then, by Theorem 36, $f^{\mathcal{M}}(s, s') = 0 = G(f^{\mathcal{M}})(s, s')$. On the other hand, if $s \not\sim_{\mathcal{M}} s'$, $f^{\mathcal{M}}(s, s') = F(f^{\mathcal{M}})(s, s') = G(f^{\mathcal{M}})(s, s')$. \square

Proof (of Lemma 37). In [10] it has been shown that deciding probabilistic bisimilarity over an MC is **P**-hard. We proceed showing that the problem of computing bisimilarity on an SMM \mathcal{M} can be turned to the problem of computing probabilistic bisimilarity on an MC \mathcal{M}' .

Recall that the total variation distance is a metric over $\Delta(\mathbb{R}_+)$, therefore, given two states $s, s' \notin A$, the problem of checking $\rho(s) = \rho(s')$ corresponds to verify $\|\rho(s) - \rho(s')\|_{\text{TV}} = 0$. Let X be a set of labels disjoint from AP , and let $l: S \setminus A \rightarrow X$ be a map such that $l(s) = l(s')$ iff $\rho(s) = \rho(s')$. By hypothesis, l can be constructed in polynomial time in $\text{size}(\mathcal{M})$.

Let $\mathcal{M}' = (S, L, \tau', \ell')$ be an MC defined as follows. The set of labels L is $2^{AP \cup X}$; the transition probability $\tau': S \rightarrow \mathcal{D}(S)$ is defined as $\tau'(s) = \tau(s)$ if $s \notin A$, and $\tau'(s) = \chi_{\{s\}}$ if $s \in A$; the labeling function $\ell': S \rightarrow L$ is defined as $\ell'(s) = \ell(s) \cup \{l(s)\}$ if $s \notin A$, and $\ell'(s) = \ell(s)$ if $s \in A$.

Now we show that any bisimulation $R \subseteq S \times S$ for \mathcal{M}' in the sense of [10] is also a bisimulation for \mathcal{M} in the sense of Definition 4. Let R be a bisimulation for \mathcal{M}' and assume $s R s'$. By [10] $\ell'(s) = \ell'(s')$ and $\forall C \in S/R. \tau'(s)(C) = \tau'(s')(C)$. Clearly, $\ell'(s) = \ell'(s')$ implies that $\ell(s) = \ell(s')$ and either $s, s' \in A$ or $s, s' \notin A$ and $\rho(s) = \rho(s')$. If $s, s' \notin A$, $\forall C \in S/R. \tau'(s)(C) = \tau'(s')(C)$ also implies that $\forall C \in S/R. \tau(s)(C) = \tau(s')(C)$, by construction of τ' . \square

Proof (of Theorem 38). The proof uses the same idea of [10]. Proceed by showing that $\vartheta^{\mathcal{M}}$ can be characterized as the solution of a linear program that can be solved in polynomial time in $\text{size}(\mathcal{M})$.

By Lemmas 35 and 44, $\vartheta^{\mathcal{M}}$ is the unique fixed point of $G: [0, 1]^{S \times S} \rightarrow [0, 1]^{S \times S}$. Thus, by Tarski's fixed point theorem we have that

$$\vartheta^{\mathcal{M}} = \bigsqcup \{d \in [0, 1]^{S \times S} \mid d \sqsubseteq G(d)\}.$$

This allows us to characterize $\vartheta^{\mathcal{M}}$ as the solution of the following linear program

$$\begin{array}{ll} \text{maximize} & \sum_{u, v \in S} d_{u, v} \\ \text{such that} & d_{u, v} = 0 \quad u \sim_{\mathcal{M}} v \\ & d_{u, v} = 1 \quad u \not\sim v \\ & \alpha_{u, v} = \|\rho(u) - \rho(v)\|_{\text{TV}} \quad u \equiv v \text{ and } u, v \notin A \\ & d_{u, v} \leq \alpha_{u, v} + (1 - \alpha_{u, v}) \sum_{x, y \in S} d_{x, y} \cdot \omega_{x, y} \quad u \equiv v \text{ and } u, v \notin A, \\ & \omega \in \Omega_{u, v}, u \not\sim_{\mathcal{M}} v \end{array}$$

where, for arbitrary states $s, s' \notin A$, $\Omega_{s,s'}$ denotes the set of vertices of the transportation polytope $\Omega(\tau(s), \tau(s'))$ described by the following linear constraints

$$\begin{aligned} \sum_{u \in S} \omega_{u,v} &= \tau(s)(u) & u \in S \\ \sum_{v \in S} \omega_{u,v} &= \tau(s')(u) & v \in S \\ \omega_{u,v} &\geq 0 & u, v \in S \end{aligned}$$

Indeed, as noticed in [10], for each fixed \mathbf{d} the linear function mapping a feasible ω to $\sum_{u,v} d_{u,v} \cdot \omega_{u,v}$ achieves its minimum on $\Omega(\tau(s), \tau(s'))$ as some vertex. Thus, the (finite) set of vertices $\Omega_{u,v}$ suffices to describe the same feasible region obtained by using the (infinite) set of $\Omega(\tau(s), \tau(s'))$.

As showed in [10], even though the above linear program may have exponentially many constraints in the number of states ($\Omega_{u,v}$ may have exponentially many elements), it admits a polynomial-time separation algorithm based on the computation of a transportation problem (see [10, Proposition 20]) that, given any instance of the variables d , it can check for its feasibility in polynomial time and, whether it is not feasible, it returns one of the constraints that is not satisfied. This permits to solve the above linear program in polynomial time in $size(\mathcal{M})$ using the ellipsoid method.[26]. \square

B Transportation Problem

In 1941 Hitchcock and, independently, in 1947 Koopmans considered the problem which is usually referred to as the (homogeneous) *transportation problem*. This problem can be intuitively described as: a homogeneous product is to be shipped in the amounts a_1, \dots, a_m respectively, from each of m shipping *origins* and received in amounts b_1, \dots, b_n respectively, by each of n shipping *destinations*. The cost of shipping a unit amount from the i -th origin to the j -th destination is $c_{i,j}$ and is known for all combinations (i, j) . The problem is to determine an optimal *shipping schedule*, i.e. the amount $x_{i,j}$ to be shipped over all routes (i, j) , which minimizes the total cost of transportation.

It can be easily formalized as a linear programming problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^m \sum_{j=1}^n c_{i,j} \cdot x_{i,j} \\ \text{such that} \quad & \sum_{j=1}^n x_{i,j} = a_i & (i = 1, \dots, m) \\ & \sum_{i=1}^m x_{i,j} = b_j & (j = 1, \dots, n) \\ & x_{i,j} \geq 0 & (i = 1, \dots, m \text{ and } j = 1, \dots, n) \end{aligned}$$

The set of schedules feasible for a transportation problem, which is formalized as a conjunction of linear constraints, describes a (bounded) convex polytope in \mathbb{R}_2 , often called transportation polytope.

There are several algorithms in literature which efficiently solve (not necessarily homogeneous) transportation problems. Among these we recall [14,18].